NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

MODELS FOR MULTI-ITEM INVENTORY SYSTEMS
WITH CONSTRAINTS

by

William Edward Daeschner, Jr.

June 1975

Thesis Advisor: F.R. Richards

Approved for public release; distribution unlimited.

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current use are found to be incompatible with budget-constrained financial control mechanisms. A model and solution procedure are furnished to guide the line-item inventory control subsystem in exercising adaptive control of a multi-item inventory with stochastic demands and fixed resources provided in successive fiscal periods. An interactive FORTRAN computer program is furnished which exploits the Generalized Lagrange Multiplier procedure in obtaining optimal or near-optimal integer solutions to a class of non-linear integer programming problems.

Models for Multi-Item Inventory Systems With Constraints

by

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Constrained multi-item inventory systems with stochastic demands are addressed. The concept of Bad Buys is introduced and their temporal development is studied through a model which links the financial and line-item inventory control subsystems. The interaction of budget constraints and control policies having stochastic resource requirement are examined using a queueing model. Some line-item control systems in current use are shown to be incompatible with budget constrained financial control mechanisms. A model and solution procedure are furnished to guide the line-item inventory control subsystem in exercising adaptive control of a multi-item inventory with stochastic demands and fixed resources provided in successive fiscal periods. An interactive FORTRAN computer program is furnished which exploits the Generalized Lagrange Multiplier procedure in obtaining optimal or near-optimal integer solutions to a class of non-linear integer programming problems.

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I. INTRODUCTION

This paper deals with multi-item inventory systems operating with stochastic demands and subject to resource constraints. The theoretical setting is the field of Inventory Theory, which considers ways of answering the questions of whether to carry a stock of some good; if so, when to obtain or replenish the stock and how much to obtain. Operating policies are called continuous review policies if replenishments may be initiated at any time epoch at which prescribed conditions are met. Periodic review policies, in contrast, can initiate replenishments only at specified epochs of time and then when specified conditions are met. The problem addressed herein is that of operation of a multi-item, dynamic (in time as well as among items) inventory system with performance objectives and interactive competition among line items for limited resources. The principal resource limitation considered is the procurement budget, although the inventory control model for line-item control incorporates a workload constraint as well.

Multi-item inventory models have been available for some years; see for example, Hadley and Whitin [1], Vienott [2], Schrady and Choe [3], and Muckstadt [4]. Most previous work attempts a variety of cost minimization considering expected values of steady-state variable costs associated with shortage costs, ordering costs and storage costs. In

such problem formulations, the expected procurement cost per time period is independent of the decision variables and is therefore dropped from the cost expression prior to optimization. A different steady-state approach by Schrady and Choe [3] led to continuous review policies designed to minimize approximately the time-weighted shortages per unit time subject to constraints on expected number of orders placed per unit time and on the priced-out value of the total expected on-hand inventory. The Navy Retail Supply System uses a multi-item model, called the Variable Operating and Safety Level (VOSL) Model, which is based on a multi-item system described in Prichard and Eagle [5]. The available multi-item inventory models generally require stochastic amounts of resources for implementation, do not address the interaction between the procurement cost of their implementation and the constraint which may have been placed on such costs, and generally permit their optimal decision variables to assume non-integer and hence unrealizable values.

In an attempt to explain and overcome serious difficulties experienced in attempting to implement available continuous-review multi-item inventory models in inventory systems with constraints, the author introduces the notion of an Integrated Inventory Control System (IICS), providing a conceptual framework for this paper.

The concept of Bad Buys is introduced and defined operationally in the context of a multi-item inventory system

subject to budget constraints. The stochastic nature of the development of Bad Buys over time is then addressed.

A dichotomization of inventory policies based on the nature of the process determining resource consumption is made by the author. On the one hand are those policies whose consumption of resources is stochastic (Type-S policies) and on the other hand those with resource consumption well determined at the time of the allocation decision (Type-D policies). Most of the inventory policies in the literature have stochastic resource consumption when demands are stochastic but do not consider explicitly the interaction between the resources needed to operate with the policies and the resources actually made available. In this paper the author examines the impact of various budget policies on the capability of inventory systems to execute policies with stochastic resource requirements.

The author develops a model which allows inventory managers to consider a variety of demand distributions and objective functions in exercising dynamic control over a multi-item inventory. The model and proposed procedure for using the model permit consideration of the real-time inventory position of each line-item candidate for replenishment and thus account for inter-item competition for scarce replenishment resources while not requiring steady-state assumptions. Explicit consideration of the procurement budget available circumvents the problems inherent in the

interaction of constraints on procurement funds with the stochastic funding needs of common classes of replenishment policies.

An interactive computer program has been developed which is easy to use and which obtains optimal or near-optimal line-item allocations using the model developed herein for allocation to line items. It solves a class of non-linear integer programming problems quickly and permits selection of any of a set of objective functions by means of a parameter vector.

Hadley and Whitin [6] observed:

> " ... when an absolute budget constraint is imposed on a year to year basis, a steady-state model is no longer suitable in general. This is even more clear where the budget is specified and may vary from year to year. An entirely different and exceedingly complicated model is required."

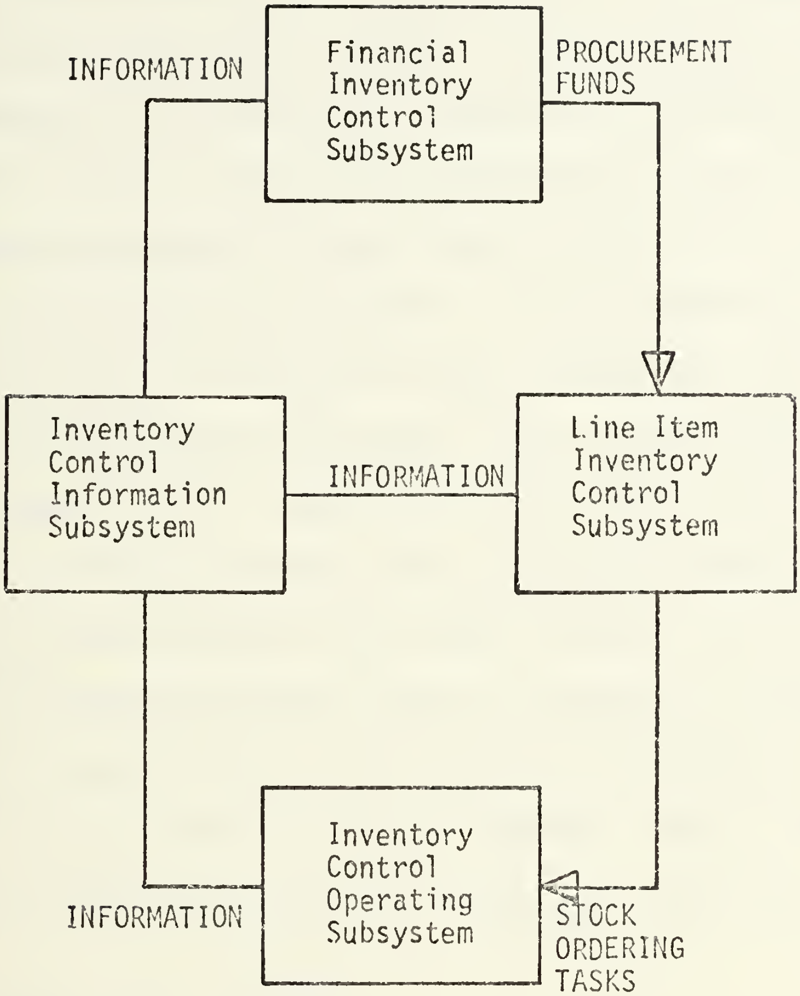
and again, in their text [1]:

> "Perhaps the most important real world constraints are budgetary restrictions on the amount that can be spent in procurements. ... However, as we have noted previously, there is no simple way of including such budgetary constraints in a model."

It is believed that progress in optimization theory now permits such real world constraints to be incorporated in inventory operating systems, through conversion of Type-S policies to Type-D policies.

A. THE INTEGRATED INVENTORY CONTROL SYSTEM (IICS)

By the acronym IICS is meant a system comprised of four interacting subsystems, displayed diagrammatically in Figure (1).



INTEGRATED INVENTORY CONTROL SYSTEM

FIGURE 1

The first is likely to be located highest in the hierarchy of the organization; it will be called the financial inventory control subsystem (FICS). FICS operates principally on aggregate or macroscopic information obtained from the inventory control information subsystem (ICIS) in order to provide periodically fixed budgets to the line item inventory control subsystem (LIICS). LIICS operates principally on line item or microscopic data contained in the ICIS, and allocates available resources, such as the stock fund procurement budget or the transaction processing capability of the system, so as to convert these available resources into stocks of the various line items carried. The inventory control operating system (ICOS) executes the procurement decisions made by the LIICS subject to constraints on capabilities, while the ICIS receives, stores, processes and provides information to the other subsystems.

B. CONTENTS AND SUMMARY

Chapter II provides a summary of historical development of selected aspects of the Navy Retail Supply System as well as a brief description of the present system. Problems experienced in operating the line-item inventory control subsystem of the Navy Retail Supply System are noted together with several adaptations of the formal policy which have been attempted by field activities in attempts to stay within budget constraints.

Chapter III introduces the concept of Bad Buys within a budget-constrained inventory system and addresses the stochastic nature of their development over time.

Chapter IV addresses the interactions between the budget policies of the FICS and the operation of a LIICS using policies with stochastic resource requirements. An analogy is presented between aspects of the FICS-LIICS interaction and entities in a class of queueing models.

Chapter V presents a model and a solution procedure which is designed to permit line-item inventory control in a multi-item context where resources are constrained. Both procurement budget and replenishment workload constraints are considered with a capability provided to allow the user to select from a class of objective functions. An interactive computer program which provides a basis for implementation of the proposed LIICS system is provided.

Chapter VI provides a model to assist the LIICS in determining how much of the available procurement budget to spend at a given replenishment epoch. Information from the line-item allocation procedure of Chapter V is used, together with information on time preference over replenishment epochs, to produce a spending plan for the remainder of the fiscal period. The spending plan is a projection, for each intended replenishment epoch of the fiscal period, of the amount of budget to be allocated, the value of the

effectiveness criterion to be achieved, and the value of the shadow price of the procurement budget.

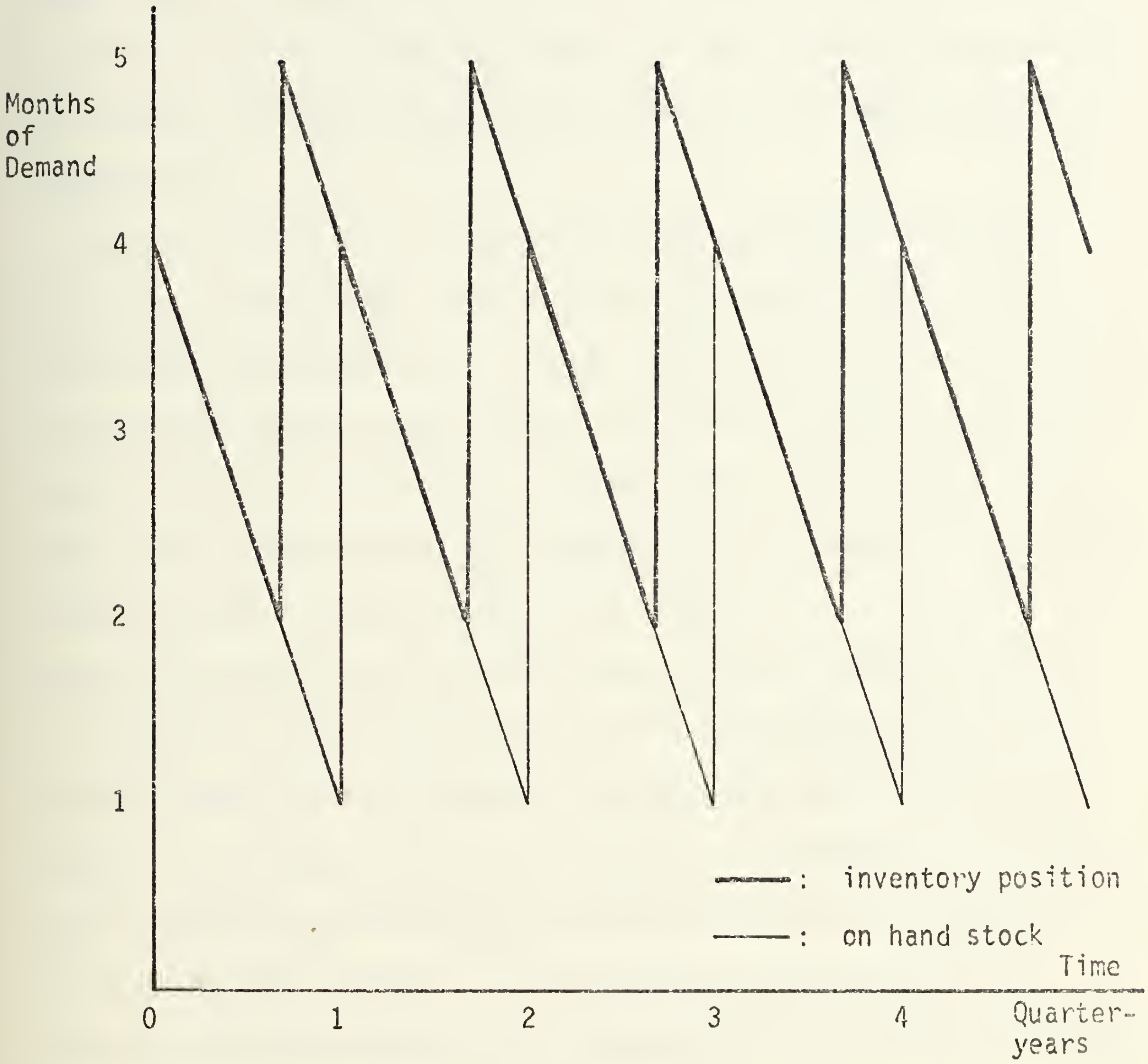
Chapter VII presents conclusions and suggestions for further research.

II. BACKGROUND

In this chapter we review developments leading to the current Navy Retail Supply System, attributes of which we have abstracted to construct the IICS model described in Chapter I. Problems observed within this system are reviewed as they relate to interactions with the procurement budgets provided by a FICS to a LIICS which attempts to operate using prescribed Type-S line-item inventory control policies.

A. HISTORICAL DEVELOPMENT

The basic financial reference model, predating the Navy Retail Supply System, is depicted in Figure (2). The inventory position is defined as the stock on hand plus stock on order less any backorders. Drawn in bold face, it coincides in this model with the on-hand inventory when no orders are outstanding. The stock on hand, represented by a lightly drawn line, diverges from the inventory position from the time an order is placed until it arrives. In Figure (2) the inventory is viewed as experiencing continuous demand at a constant rate, causing the stock on hand to vary from a "stockage objective" of four month's demand to a "safety level" (provided in recognition that demand and lead time are not so regular in practice) of one month's demands. Stocks are reordered at the time when the stock



BASIC FINANCIAL REFERENCE MODEL

FIGURE 2

on hand is just sufficient to carry the system until the time when the safety level is hit -- in our figure a lead time of one month from placement of the order to its receipt is assumed so that the reorder level is two months of stock consumption.

In the 1950s U.S. Navy line-item inventory control practices were based on the model of reality previously presented as Figure (2). Stock control clerks would estimate the monthly demand and reorder whenever the inventory position fell to or below the reorder point. The price of an item and attributes of its demand pattern other than average demand rate affected neither the decision as to when to place an order for a carried item nor how much to order.

Pressure from the Office of the Secretary of Defense forced adoption of ordering policies using economic principles. This gave rise to a number of optimization models with variable replenishment values based on the economic-order-quantity concept, as described in reference (1) wherein order quantities are determined principally from a balancing of ordering and holding costs. Assuming simple cost relationships, order quantity expressions such as

$$Q_W = k_1\sqrt{\frac{\lambda}{c}}$$

where λ is the mean rate of demand, c is the per-unit cost,

and k_1 is a cost parameter depending on both holding and ordering costs, were derived by minimizing average annual costs.

Thus, rather than always ordering three months demand, the amount to be ordered would be proportional to the square root of the ratio of the estimated demand rate to unit price.

Consideration of the variability of demand in a lead time led to formulas for determining reorder points r by formulae such as

$$r = F^{-1}\left(\frac{k_2}{k_2 + k_3}\right)$$

where k_2 is the per-unit cost incurred by overage, k_3 is the per-unit cost incurred by shortage of stock, and F^{-1} is the inverse function of the cumulative distribution for demand in a lead time. Operating under such policies requires stochastic quantities of procurement funds.

Such modeling attempted unconstrained minimization of costs which were more of the economic or conceptual variety than they were actual accounting costs. That is, the holding costs, shortage costs or order costs contributing to the cost expression which was minimized were costs which did not appear directly in the accounting systems servicing the inventory systems. It was considered, perhaps, that a system which could operate under the old, obviously-nonoptimal policies could *a fortiori* operate if the inventory policies

were optimal in a cost-minimization sense. These policies were intended to operate in a "continuous-review" environment in which stock status is closely monitored so that the receipt of demands immediately initiates a stock replenishment as soon as the inventory position falls to the reorder point. As these single-item models were implemented within multi-item inventory systems there was no accounting for interactive competition among different line-items for constrained resources.

The need to live with tight constraints on asset value led to the adoption of a multi-item inventory system, the Variable Operating and Safety Level System (VOSL). Underlying the Economic-Order-Quantity models, VOSL permits management to specify, in terms of months of demand, a range of operating and safety levels based on individual line item characteristics while holding the aggregate values within an overall specified limitation. This limitation, based on a target inventory to sales ratio, is stated as a limitation of investment in inventory expressed in terms of months of sales. This model exploited a principle known to mathematical optimizers as the principle of constraint relaxation. Rather than to require that each line item be managed to attain the desired overall ratio of inventory to sales, flexibility was granted to permit some items to have high stock turnover and other items to have a low stock

turnover while maintaining the aggregate constraint for the inventory as a whole.

A broad description of the Navy Retail Supply System from the point of view of the financial manager is provided by Eckelberger [7]. Out of a universe of some 900,000 different line items found in the 1972 Navy Management Data List as potential candidates for stockage, some 300,000 line items were carried in stocks of the Navy Retail Supply System; the stocks were physically located at some 200 locations worldwide. Sales of these carried items accounted for some 83 per cent of the total retail system sales, of annual value on the order of one-half billion dollars. An additional $100 million of transactions for not-carried items were processed financially by this system in fiscal year 1971. Control of budgets for procurement of stocks is exercised by the Navy Fleet Material Support Office, Mechanicsburg, Pa., through the monitoring and use of dollar-value summary information obtained from a financial accounting system, which operates in parallel with the line item information system used for stock control.

The basic policy directive for the Navy Retail Supply System is the Navy Fleet Material Support Office Publication "Navy Retail Management," cited as Ref. [8]. The basic VOSL model is described in Refs. [9] and [10], and a detailed look at the computer implementation of the model may be

obtained from Ref. [11], which describes the computer processing routine at a particular location.

Examination of a sample of 100 line items' computerized records, obtained from a January 1973 extract from a Naval Air Station serves to indicate some of the flavor of the attributes of the line items carried in the Navy Retail Supply System. Representative statistics are listed in Table (I). Note that the ratio of the greatest to the least value of unit price from this small sample is more than 10,000:1. It appears that there may be significant unanticipated effect from using inventory models which treat order quantity as a continuous variable, permitting order quantities such as .38592 magnetrons, rather than restricting attention to feasible integer order quantities. The range of the then-current estimates of demand per quarter varied by about a factor of about 1750:1, while the range in estimated standard deviation of demand in a quarter varied by a ratio of over 800:1. The estimated coefficient of variation ranged over values in a much smaller interval about 30:1 was the factor of maximum to minimum.

The picture portrayed in our sample is not dissimilar to that reported by the Material Readiness Index System (MARIS) Study Group [12] in which data on prices was obtained from the Ship's Parts Control Center, and is displayed in Table (II). Here, although we speak of technical material which is outside the Navy Retail Supply System, we find

TABLE I

HETEROGENEITY OF LINE ITEMS IN NAVY RETAIL SUPPLY SYSTEM

Attribute:	Min:	Max:	Max/Min
Unit prices	$ .04	$ 540.75	> 13,500
Est. quarterly mean demand	.48	841.6	> 1,750
Est. value of annual demand	3.36	20,958.02	> 6,200
Est. standard deviation of demand in a quarter	.40	329.49	> 800
Est. coefficient of variation of demand in a quarter	.0738	2.26	> 30

Average value of coefficient of variation estimate: 1.2

Data source: sample of 100 line items in VOSL inventory segment from January 1973 extract from Naval Air Station, Point Mugu.

TABLE II

SKEWNESS OF UNIT PRICES OF NAVY TECHNICAL LINE ITEMS

Cognizance Symbol (a set identifier)	Mean Price	Median Price	Number of Line Items
1H	$ 167.20	$ 13.66	122,461
2H	884.82	150.00	52,108
6H	1,192.25	467.50	4,012
2Y	96.07	23.80	4,167
1A	28.41	5.22	10,453
2A	285.04	8.40	64,592
6A	94.57	10.00	20,612

Data source: MARIS Study Group Interim Report

The ratio of mean to median unit price among these groups falls between about 3 and 25, with median value of the ratio about 9.

considerable skewness in the unit prices, with mean prices generally far above median prices in large groups of line items of the same "cognizance" class which should have some similarity of use.

In contrast to most commercial inventory systems, the Navy Retail Supply System contains an extraordinary range of types of material, from rags to sophisticated electronic parts, from hardware to the ubiquitous U.S. Government ball-point pen. Although it contains items with demand patterns suggesting seasonal demands and decentralized items where the attainable procurement unit price might not be independent of the quantity ordered (and so on), the sheer size of the system, relatively high degree of automation and attendant low level of available skilled human stock control effort per line item have tended to generate uniform procedures which might work best on the average but not so well in each individual instance.

A retailer such as a supermarket or Sears, Roebuck & Co., likes to have relatively low inventories of stocks with high stock turnover rates. Those systems typically face short lead times for resupply, with many substitutes available for each individual line item. The commercial retailer can clear its stock of excesses and recoup its direct investment costs by placing stock on sale. If "business is good" and demand schedules should rise, pricing and advertising policy can be used to turn demand schedule

increases into increased profits. The value of a unit of a line item of stock is very closely related to the expected increment to profits obtained by the individual unit — given a markup policy the value of a unit might be considered to be proportional to the cost to a close approximation.

The scale and scope of military inventory systems typically dwarf those of civilian enterprises. Many line items are highly technological and have no satisfactory substitutes. Replenishment lead times often are long and highly unpredictable. Excessing actions generally return far less than the direct investment costs and then only after long delays. The stimulation of consumption of slow-moving stock by marketing appeals is generally considered undesirable and therefore this management option is foreclosed. Pricing policies are not used as a tool to enhance the financial viability of the inventory system. Significant increase in the quantity of goods demanded may likely result from war or other armed conflict; hence being caught with stocks "too low" may have profoundly adverse consequences. The differences between military and commercial retail inventory systems would seem to justify larger inventories in the military case (with lower stock turn-over rates) than in the commercial case.

B. PROBLEMS INVOLVING TYPE-S POLICIES AND FIXED BUDGETS

After a decade of experience using the VOSL system, several recurring difficulties of "living with" the system

have been noted. We feel that the difficulty experienced derives from the implementation of a Type-S policy within a budget-constrained system. In a personnel communication the Requirements Division (i.e., Stock Control) Director of a major overseas stock point stated, "Over a period of time it became obvious that the VOSL Model was going to generate buys which exceeded our budget constraint... ." At that stock point an auxiliary criterion was developed to decide on a priority basis which of those triggered replenishments were to be released and which were to be ignored. Personal conversations with officers at other activities have revealed a variety of responses to this common dilemma of having insufficient procurement funds to implement the VOSL policy. At a Naval Supply Center the order quantities of triggered replenishments were decreased by a fraction determined so that the cost of releasing the indicated (modified) replenishments did not exceed the available budget. At a Naval Air Station, the VOSL orders were released without modification until no more funds were available, whereupon no further replenishments were released until procurement funds became available in the next fiscal quarter. Lewis and Perkins [13] report four other ordering policy alterations:

1. Place a monetary limit of ten dollars on resupply requisitions.

2. Set order quantities to be equal to 30 days of forecast demand.

3. Set reorder points to zero.

4. Modify stocking and replenishment criteria to stock and order only those items having at least six requisition demands in six months.

It is apparent that there is no consistent practice for inventory control of items which are nominally managed by the VOSL system. This, it is urged, is not due to capricious and arbitrary disregard of established policies. It is, rather, the natural result of the imposition of financial and stock control policies which cannot be simultaneously followed in a very complex system which is "supposed to work." Since failure to implement duly established policies is considered to reflect unfavorably upon an officer, who is held accountable for his actions (and inspected by the Inspector General for adherence to established policies), the sharing of information on common problems has taken place, so far as it can be determined, tentatively and across informal, horizontal lines of communication.

There are two widely separated views concerning the source or cause of the inability to accomplish the implementation of VOSL within its intended organizational milieu. On the one side, those with a financial management orientation argue that funds are granted on a sales replacement basis so that an efficiently functioning stock control branch should be able to operate within the funding limitations — if only the stock controllers would cease and desist from investing in "dead stock." Those with the stock control

orientation contend that funds are never available in quantity adequate to service the vagaries of demand and that large increases in the quantities demanded of some items, increases in lead time, inflation and bad buys of stock are ignored by the financial manager's view of his aggregate statistics. Murphy's Law is seen to reign supreme over the stock replenishment process with demand pausing after reorders are placed for one item (or else exceeding the stock on hand) while another item, long dormant, receives an unexpected flurry of demand.

Couched in engineering jargon, the financial manager, dealing in aggregates, sees a relatively favorable signal-to-noise ratio in the value of demands experienced in successive periods. The stock controller, making the individual replenishment decisions, sees "with a microscope" almost everywhere a relatively unfavorable signal-to-noise ratio in the quantity demanded for a line item from one period to the next.

In this chapter the development of the Navy Retail Supply System has been summarized, the heterogeneity of line items managed in this system has been noted and inherent differences between military inventory systems and commercial inventory systems have been noted. Inherent difficulties occasioned by the interaction of Type-S policies within a LIICS which must operate within specified constraints on procurement funds were noted as were several informal alterations of official policies which have become known to the author.

III. THE DEFINITION OF BAD BUYS AND THEIR STOCHASTIC BEHAVIOR OVER TIME

In this chapter the concept of Bad Buys is introduced, mathematically defined, and two examples are presented to illustrate their behavior over time. A third example relates the management option of disposal action to the concept of Bad Buys within a multi-item inventory system with asset constraints.

When multi-item inventories are subjected to random demands it frequently happens that one line item may have what appears to be an excessive quantity of stock on hand, reflecting Bad Buys, while simultaneously another line item is out of stock with unsatisfied demands either being back-ordered or else resulting in lost opportunities for issue. Such an inventory may have considerable stock assets with great dollar value, but the maldistribution of the asset value among the various line items decreases the capability of the inventory system to accomplish its objectives. Of course, after the random demands have been experienced it may be easy to determine what distribution of assets among the various line items might have precluded the out-of-stock conditions.

The operation of the LIICS policy is seen as a transformation of usable resources into stocks of dissimilar line items so as to maintain a desirable balance of assets over

the entire range of the inventory of carried line items within the limits of resources provided. Certain factors are considered which tend to affect the presence of Bad Buys in an inventory.

A. BAD BUYS DEFINED

Consider a multi-item inventory system containing N line items. At succeeding epochs of time, $t_p, t_{p+1}, \ldots,$ orders are placed and simultaneously received. Let $\underline{b}_p$ be the N component legacy vector of inventory on hand at time $(t_p)^-$ while $\underline{x}_p$ denotes the corresponding vector of inventory on hand at time $(t_p)^+$ just after any stock orders have been placed and received. Let A_p be the total dollar value of stock assets permitted in the inventory at time $(t_p)^+$, $A_p > 0$, and let B_p be the budget available for procurement of stock in the period beginning at time t_p, made available at time t_p. Let c_p be an N-vector of unit prices c_{jp} of line item j at time t_p. Let $z_p(\underline{x}_p)$ be the objective function, and consider the following two one-period optimization problems (P1) and (P2) (suppressing p):

$$
\text{(P1)} \qquad
\begin{aligned}
& \max \quad z(\underline{x}) \\
\text{s.t.} \quad & \underline{c}^T(\underline{x} - \underline{b}) \leq B \\
& x_j - b_j \geq 0 \text{ for all } j \\
& x_j,\ b_j \ \text{nonnegative integers}
\end{aligned}
$$

with optimal solution $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_N^*)$

and

$$\text{(P2)} \qquad \begin{array}{ll} & \max \quad z(\underline{x}) \\ \text{s.t.} & \underline{c}^T\underline{x} \leq B + \underline{c}^T\underline{b} = A \\ & x_j,\ b_j \text{ nonnegative integers} \\ & \text{with optimal solution } \underline{x}^{**} \end{array}$$

The relaxed problem (P2) is obtained from (P1) by providing the option of converting into dollars without loss any legacy of stock brought forward at the time being considered -- and then reallocating the value of such converted legacy optimally.

Every feasible solution to (P1) is also a feasible solution to (P2), but since x_k^{**} may be less than the legacy b_k for some item, the converse is not true. Thus,

$$z(\underline{x}^*) \leq z(\underline{x}^{**}).$$

We introduce a measure of the effect of Bad Buys $L(\underline{b})$

$$\boxed{L(\underline{b}) = z(\underline{x}^{**}) - z(\underline{x}^*)}$$

where it is understood that the optimal solution of (P1) is a function of the starting legacy vector $\underline{b}$. $L(\underline{b})$ gives the value of the loss in objective function units due to the maldistribution of the legacy of inventory brought

forward; i.e. the loss due to the "inherited" bad buys. Thus, if L is positive we say that a budget-constrained inventory system has Bad Buys. In a multi-item inventory system with random demands, the appearance and disappearance of Bad Buys are stochastic events which depend on the actions and interactions of the several components of the integrated inventory control system.

Definition. Let $B = \{\underline{b}: L(\underline{b}) > 0\}$. Then a vector $\underline{b} \in B$ is defined to be a Bad Buy, and B is the set of all Bad Buy vectors at the time epoch $(t_p)^-$.

It is noted that $L(\underline{b})$ is defined in terms of the difference in objective function value of two related optimization programs; thus $L(\underline{b})$ is implicitly a function of the number of line items, their demand distributions and cost coefficients, the choice of objective function and of the asset constraint, A.

In the inventory model, the fixed or allocated assets might be stocks of different line items, stocks of finished goods, or even of personnel with different skills. The assets subject to allocation might be dollars, raw materials or pools of untrained people.

B. TEMPORAL DEVELOPMENT OF BAD BUYS

Let D_{jp} be the random demand for item j in period p, and suppose that a budget is provided by the FICS at the beginning

of each time period. In the examples it is assumed that the asset value determined by the FICS is the expected value of the (known) demand distribution for each item multiplied by the corresponding unit prices summed over all items and multiplied by a constant k reflecting the desired inventory/sales ratio. That is, the asset value for period p is

$$A_p = k \sum_{j=1}^{N} c_{jp} \; E(D_{jp}), \qquad k > 0$$

The budget increment for period p is then determined as the difference between the asset value A_p and the value of the inventory legacy, or zero, whichever is greater.

$$B_p = \max \{0, A_p - \sum_{j=1}^{N} c_{jp} \; b_{jp}\}$$

This amount is provided to the LIICS as a procurement budget for period p.

In the examples which follow, it is assumed that the objective of the LIICS is to maximize the expected value of sales from stocks available in the period at hand. Since the value of demand is equal to the sales S plus the lost sales Y, the maximization of expected sales is equivalent to the minimization of expected lost sales. The latter objective is more commonly stated but the maximization orientation is selected here. For simplicity, it is assumed in the examples that leadtimes are zero and that demands

in excess of available stock are lost. Further it is assumed that the probability distributions of demand are independent among items and over the different time periods. Later, when a solution algorithm is developed, these assumptions are relaxed. Random demands are incurred from a known probability distribution. The inventory carried forward to period p+1 from period p is determined by

$$b_{j,p+1} = \max\,(0, x^*_{jp} - d_{jp})$$

where d_{jp} is the observed demand for item j in period p. The LIICS solves (P1) with objective function $z_p(\underline{x}_p)$ given by

$$z_p(\underline{x}_p) = \sum_{j=0}^{N} c_{jp} \left(\sum_{i=1}^{x_{jp}} iP(D_{jp} = i) + x_{jp}P(D_{jp} > x_{jp}) \right)$$

Three examples are presented to illustrate the solution of (P1) and (P2). To solve more realistic problems economically for inventories containing upwards of ten thousand line items, an efficient solution algorithm is required. Such a procedure for obtaining near optimal integer allocations is presented in Chapter V. It adapts the Generalized Lagrange Multiplier (GLM) method of Everett [14] to a class of inventory problems.

1. Example One

Consider an inventory system consisting of three line items with legacy vector, cost vector and demand

distribution given in Table III. Total assets A are limited to \$48 in each period. For simplicity of notation we suppress p where feasible.

$\underline{b}_1^T = (0,5,2)$
$\underline{c}^T = (3,4,12)$

Prob.	i	0	1	2	3	4	5	$E(D_j)$
$P(D_1=i)$		0	0	0	1/3	1/3	1/3	4
$P(D_2=i)$		1/4	0	0	0	3/4	0	3
$P(D_3=i)$		1/3	1/3	1/3	0	0	0	1

PARAMETERS FOR BAD BUY EXAMPLE ONE

TABLE III

First, considering the relaxed problem (P2) with free and ready conversion of stock assets into dollars, we solve

$$\max_{\underline{x}} \quad z(\underline{x}) = 3\left(\sum_{i=1}^{x_1} iP(D_1=i) + x_1P(D_1 > x_1)\right)$$
$$+ 4\left(\sum_{i=1}^{x_2} iP(D_2=i) + x_2P(D_2 > x_2)\right)$$
$$+12\left(\sum_{i=1}^{x_3} iP(D_3=i) + x_3P(D_3 > x_3)\right)$$

subject to

$$3x_1 + 4x_2 + 12x_3 \leq 48$$

$$x_1, x_2, x_3 \in \{0,1,2, \ldots\}$$

In solving this problem using dynamic programming [15], stages correspond to the three line items and the state of the system is summarized by the assets remaining to be allocated. The state transformation equation is

$$s_{n-1} = s_n - c_n x_n$$

while the recursive equation is

$$f_n(s_n) = \max_{x_n} \{c_n(\sum_{i=1}^{x_n} iP(D_n=i) + x_n P(D_n > x_n)) + f_{n-1}(s_n - c_n x_n)\}.$$

Adopting the convention, as is done throughout this chapter, that in case of alternate optima, the lexicographically smaller [16] is selected, the optimal solution to (P2), for all p, is found to be $\underline{x}_p^{**} = (4,3,2)$ with optimal value $z(\underline{x}_p^{**}) = \$32$.

Next the legacy vector $\underline{b}_1$ with committed assets of $0(3) + 5(4) + 2(12) = \$44$ is used to solve the restricted problem (P1):

$$\max_{\underline{x}_1} \quad z(\underline{x}_1)$$

$$\text{subject to} \quad 3x_{11} + 4(x_{21}-5) + 12(x_{31}-2) \leq 48-44 = 4$$

$$x_{11} \in \{0,1,2, \ldots\}$$

$$x_{21} \in \{5,6,7, \ldots\}$$

$$x_{31} \in \{2,3,4, \ldots\}$$

We obtain $\underline{x}^* = (1,5,2)$ with value $z(\underline{x}^*) = \$27$.

Since $L(\underline{b}_1) = 32 - 27 = \$5$, Bad Buys are present and the expected loss which results from the maldistribution of the legacy in period one is \$5 over that which would have been attainable if there were no Bad Buys.

If no new Bad Buys are generated, future demands will act to "eat down" the stocks available to place the inventory system into a more favorable position. For example, demands occurring in period one will be applied against $\underline{x}_1^* = (1,5,2)$, the best inventory position attainable in light of the Bad Buys reflected by the beginning stock legacy vector. The legacy available at the beginning of period two has the following distribution:

Legacy ($\underline{b}_2$) from	Demands ($\underline{d}_1$) with	Probability
(0,1,0)	(3,4,2),(4,4,2),(5,4,2)	1/4
(0,1,1)	(3,4,1),(4,4,1),(5,4,1)	1/4
(0,1,2)	(3,4,0),(4,4,0),(5,4,0)	1/4
(0,5,0)	(3,0,2),(4,0,2),(5,0,2)	1/12
(0,5,1)	(3,0,1),(4,0,1),(5,0,1)	1/12
(0,5,2)	(3,0,0),(4,0,0),(5,0,0)	1/12

By solving problem (P1) with each of the legacy vectors $\underline{v}_2$ and comparing each such solution with the optimal solution to the relaxed problem (P2) it is determined that the Bad Buys, $\underline{b}_1$, have been subject to attrition in all but the latter event, namely the event that $\underline{b}_2 = (0,5,2)$. That is, the resulting legacy vectors at period two have $L(\underline{b}) = 0$ except for $L((0,5,2)) = \$5$. In this simple example with stationary demand from a known distribution, the attrition of Bad Buys will eventually occur, transforming the legacy vector in some period p to a state such that the optimal vector for (P1), $\underline{x}_p^*$, has the same objective function value as does $\underline{x}_p^{**}$. Thus, in this example, Bad Buys cannot be regenerated once removed by attrition, so that, given attrition, then for each k = 1,2, ... ,

$$\underline{x}_{p+k}^* = \underline{x}_{p+k}^{**}$$

and

$$z(\underline{x}_{p+k}^*) = \$32.$$

It follows from the above that in this example the optimal inventory vector $\underline{x}_p^*$ may be viewed as following a stationary Markov process (depending on the legacy vector $\underline{b}_p$) with two states:

State 1: $\underline{x}_p^{**}$ (no Bad Buys)

State 2: $\underline{x}_p^{*}$: $\underline{x}_p^{*} \neq \underline{x}_p^{**}$ (Bad Buy legacy)

In our particular example the probability transition matrix is given by:

$$
P_{ij}: \quad \begin{array}{c|cc} \text{State} & 1 & 2 \\ \hline 1 & 1 & 0 \\ 2 & 11/12 & 1/12 \end{array}
$$

In a more general case with a stationary, independent distribution for demand, unchanging unit prices and budgeting policy, the optimal inventory vector $\underline{x}_p^{*}$ can still be described as a stationary Markov process. If state one is the vector $\underline{x}_p^{**}$ (no Bad Buys) and states 2 through m correspond to the m other possible values of $\underline{x}_p^{*}$ which contain Bad Buys in at least one line item, then the Markov process $\{\underline{x}_p^{*}\}$ has an absorbing state at state 1, and all other states are transient. Without lexicographical or other preference determination over the members of the class of optimal solutions to (P2) then state 1 would need to be expanded to include all optimal solutions of (P2) as a closed, communicating class.

The corresponding steady state probabilities are readily seen to be

$$\pi_1 = \lim_{p\to\infty} P(\text{State } 1) = 1$$

$$\pi_2 = \lim_{p\to\infty} P(\text{State } 2) = 0$$

so that, as time passes, the probability that attrition removes Bad Buys approaches one. Furthermore, in this simple example, the expected sales in period p are given by

$$E(z(\underline{x}_p^*)) = \$32 - \$5(1/12)^{p-1}$$

so that the expected value of sales from stock converges geometrically to the expected valued of sales from stock in the relaxed problem (P2). In the general stationary case $E[z(\underline{x}_p^*)]$ will still converge to $z(\underline{x}^{**})$, although, perhaps, not geometrically.

2. Example Two

In example one we showed that, if demand distributions are stationary and all other system parameters remain static, Bad Buys are eventually removed by attrition. In actual practice, the situation is dynamic and Bad Buys are continually created and removed. In example two the demand distribution is nonstationary. We show that Bad Buys can be generated because of changes in the demand distributions

for the line items. Again consider an inventory system consisting of three line items with unit prices $c_{jp} = c_j$ for all p and for j = 1,2,3. Let the initial legacy be (0,0,0) so that no Bad Buys exist at the outset. We look at a horizon of four periods, and we assume that the LIICS knows exactly the demand distribution for each line item in the period at hand, but that the demand distribution is unknown before that time. The parameters and the demand distributions used in the example are summarized in Table IV, in which, for example, the probability that demand in period two for item three equals three, $P(D_{32} = 3)$, is found to be .2. The LIICS attempts to maximize expected sales in

			$P(D_{jp}=i)$							
Period	Item	c_{jp}	i=0	i=1	i=2	i=3	i=4	i=5	b_{j1}	$E(D_{jp})$
1	1	3	1/3	1/3	1/3	0	0	0	0	1
	2	3	0	0	1/2	1/2	0	0	0	5/2
$A_1=24$	3	3	1/6	1/6	1/6	1/6	1/6	1/6	0	5/2
2	1	3	1/4	1/4	1/4	1/4	0	0		3/2
$A_2=24$	2	3	1/2	0	0	0	0	1/2		5/2
	3	3	1/5	1/5	1/5	1/5	1/5	0		2
3	1	3	1/5	1/5	1/5	1/5	1/5	0		2
	2	3	1/4	1/4	0	0	1/4	1/4		5/2
$A_3=24$	3	3	1/4	1/4	1/4	1/4	0	0		3/2
4	1	3	1/6	1/6	1/6	1/6	1/6	1/6		5/2
	2	3	1/6	1/3	0	0	1/3	1/6		5/2
$A_4=24$	3	3	1/3	1/3	1/3	0	0	0		1

PARAMETERS FOR BAD BUY EXAMPLE TWO

TABLE IV

each given period subject to the constraints of the basic model. The optimal solution to the relaxed problem (P2) for each period is given in Table V .

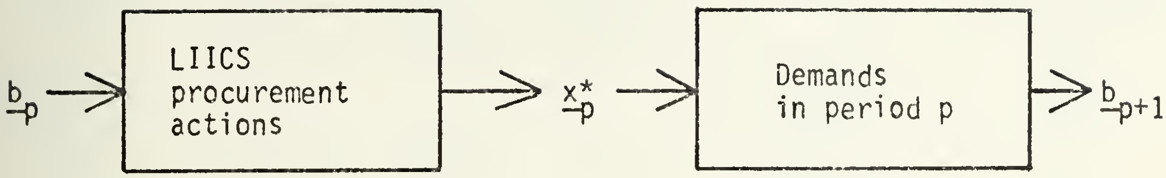
Period	$\underline{x}_p^{**}$	$z(\underline{x}_p^{**})$
1	(1,3,4)	16.50
2	(1,5,2)	13.95
3	(2,4,2)	14.70
4	(3,4,1)	15.00

SOLUTIONS TO (P2) FOR BAD BUY EXAMPLE TWO

TABLE V

Analysis proceeds throughout the four periods as in Example 1 by obtaining $\underline{x}_1^*$ from $\underline{b}_1$, considering the effect of $\underline{d}_1$ upon $\underline{x}_1^*$, obtaining a probability distribution over all possible values of $\underline{b}_2$, optimizing problem (P1) for each possible legacy $\underline{b}_2$ to obtain $\underline{x}_2^*$, and so on. We obtain a sequence of inventory vectors $\underline{x}_p^*$ (each optimal for some particular legacy vector $\underline{b}_p$) which constitutes a non-stationary Markov chain.

In each period, a legacy vector $\underline{b}_p$ is transformed into an asset vector $\underline{x}_p^*$ through the procurement actions of the LIICS. This asset vector is, in turn, transformed into a legacy vector $\underline{b}_{p+1}$ for the next period by the demands that occur in period p. This is illustrated in the input-output diagram of Figure 3.



TRANSFORMATION PROCESS FOR BAD BUY EXAMPLES

FIGURE 3

Table VI provides a summary of such optimizations over four periods. To reduce the number of entries of $\underline{b}_p$ which are mapped into a common vector $\underline{x}_p$, the letter "G" is used to represent line item legacies that did not change the vector $\underline{x}^*_p$ from that which was optimal for zero legacies of these line items. The probabilities in the last column of Table VII are the same as the transition probabilities $P(\underline{x}^*_p \rightarrow \underline{x}^*_{p+1})$, for the Markov chain $\underline{x}^*_p$, $p = 1, 2, \ldots$. For example, the legacy (G,G,3) in period two is optimized to vector $\underline{x}^*_2 = (1,4,3)$ with optimal value of objective function equal to \$13.65. This vector (1,4,3) is transformed by demands in period two either to a vector in the class (G,G,G) indicating no Bad Buys with probability .8 or else to a vector in the class (G,G,3) with probability .2. This legacy for period three of (G,G,3) is optimized to a vector, (2,3,3), in period three different from that in period two by the changed

Period	$\underline{b}_p$	$\underline{x}^*_p$	$z(\underline{x}^*_p)$	$\underline{b}_{p+1}$	$P(\underline{x}^*_p \to \underline{b}_{p+1})$
1	(0,0,0)	(1,3,4)	$16.50	(G,G,G)	2/3
				(G,G,3)	1/6
				(G,G,4)	1/6
2	(G,G,G)	(1,5,2)	13.95	(G,G,G)	1/2
				(G,5,G)	1/2
	(G,G,3)	(1,4,3)	13.65	(G,G,G)	4/5
				(G,G,3)	1/5
	(G,G,4)	(1,3,4)	12.75	(G,G,G)	3/5
				(G,G,3)	1/5
				(G,3,4)	1/10
				(G,0,4)	1/10
3	(G,G,G)	(2,4,2)	14.70	(G,G,G)	3/4
				(G,G,2)	1/4
	(G,5,G)	(2,5,1)	13.95	(G,G,G)	3/4
				(G,5,2)	1/4
	(G,G,3)	(2,3,3)	13.95	(G,G,G)	1/2
				(G,G,2)	1/4
				(G,G,3)	1/4
	(G,0,4)	(2,2,4)	12.45	(G,G,G)	1/4
				(G,G,2)	1/4
				(G,G,3)	1/4
				(G,3,4)	1/16
				(G,1,4) (G,2,4)	}3/16
	(G,3,4)	(1,3,4)	12.15	(G,G,G)	1/4
				(G,G,2)	1/4
				(G,G,3)	1/4
				(G,G,4)	1/4
4	(G,G,G)	(3,4,1)	15.00	-	-
	(G,G,2)	(2,4,2)	14.50	-	-
	(G,5,G)	(2,5,1)	14.00	-	-
	(G,G,3)	(2,3,3)	13.00	-	-
	(G,G,4)	(2,2,4)	11.50	-	-
	(G,1,4)	(2,2,4)	11.50	-	-
	(G,2,4)	(2,2,4)	11.50	-	-
	(G,3,4)	(1,3,4)	11.00	-	-

SOLUTIONS TO (P1) FOR BAD BUY EXAMPLE TWO

TABLE VI

demand distribution used in the optimization of (P1). Table VII summarizes some transient results which show the impact of the Bad Buys generated because of the changing demand process.

	p=1	p=2	p=3	p=4
$P(x^*_{1p} = x^{**}_{1p})$	1	1	.98	.72
$P(x^*_{2p} = x^{**}_{2p})$	1	.67	.57	.88
$P(x^*_{3p} = x^{**}_{3p})$	1	.67	.57	.80
$P(\underline{x}^*_p = \underline{x}^{**}_p)$	1	.67	.56	.72
$z(\underline{x}^{**}_p)$	16.50	13.95	14.70	15.00
$E(z(\underline{x}^*_p))$	16.50	13.70	14.32	14.75
$E[L(\underline{b}_p)]$	0	.25	.38	.25

TRANSIENT RESULTS (P1) FOR BAD BUY EXAMPLE TWO

TABLE VII

3. Example Three

We have considered the temporal development of Bad Buys in a multi-item inventory system with asset constraints and provided examples wherein the optimal inventory vector relative to the inventory legacy follows a Markov process. The introduction of disposal action as a management option permits an introduction of cases intermediate between problem (P1) and problem (P2).

Consider a modification of problem (P1) in which the line item inventory control subsystem has the option to

modify the stock replenishment budget B_p by disposal of any of its legacy. Let $\underline{\delta}_p$ be a decision vector where the j^{th} component of $\underline{\delta}_p$ is the number of units of line item j to be disposed of at the beginning of period p. Thus, $\delta_{jp} \varepsilon \{0,1,2, \ldots , b_{jp}\}$. The new legacy vector, after disposal, is given by $\underline{b}'_p = \underline{b}_p - \underline{\delta}_p$. Let s_{jp} be the fraction of the unit price of the j^{th} line item that can be recovered by disposal action at the beginning of period p — a salvage factor and let $\underline{c}_{s,p}$ be a vector whose components are $c_{jp}s_{jp}$. Therefore, after disposal, the budget is $B'_p = B_p + \underline{\delta}_p^T\underline{c}_{s,p}$. We assume that disposal action is immediate.

In order to decide whether or not disposal actions should be undertaken, one should look at the expected performance resulting from the modified legacy $\underline{b}'_p$ and the modified budget B'_p compared with expected performance resulting from $\underline{b}_p$ and B_p. In example 1, $\underline{b}_1 = (0,5,2)$ and $B_1 = 48 - (3,4,12)(0,5,2)^T = 4$. Consider $\underline{\delta}_1 = (0,0,1)$, yielding $\underline{b}'_1 = (0,5,1)$ and $B'_1 = \$4 + 12s_{31}$. On examination of the solution to (P1) as a function of the parameter s_{31} (assumed to lie in [0,1]) we find:

$$\begin{aligned} z'(\underline{x}^*_p) &= \$23 \text{ if } s_{31} \ \varepsilon\ [0,1/6) \\ &= \$26 \text{ if } s_{31} \ \varepsilon\ [1/6,5/12) \\ &= \$29 \text{ if } s_{31} \ \varepsilon\ [5/12,7/12) \\ &= \$31 \text{ if } s_{31} \ \varepsilon\ [7/12,5/6) \\ &= \$32 \text{ if } s_{31} \ \varepsilon\ [5/6,1.0) \end{aligned}$$

Since $z'(\underline{x}_p^*) > z(\underline{x}_p^*) = \$27$ for all values of the parameter $s_{31} \geq 5/12$, a disposal of one unit of item 3 would be justifiable if the inventory system could recoup at least five-twelfths of the purchase price. Also, as expected,

$$\lim_{s_{31} \to 1} z'(\underline{x}_p^*) = z(\underline{x}_p^{**}) = \$32.$$

C. IMPLICATIONS OF THE MODELS

This section discusses implications of the models with respect to the effect of Bad Buys on the performance of an IICS. The simple examples presented in the preceding section were selected to demonstrate how Bad Buys may be generated and how they behave over time. Other ways in which Bad Buys may be generated and removed over time are presented in this section.

From one broad perspective, Bad Buys arise from changing preferences over vectors of inventory stocks due to modifications of perceptions of the costs and effectiveness associated with these vectors. Bad Buys may arise through modification of objectives, from differential changes of the prices of various inventory line items, from fluctuation of budgets, from forced inclusion of new items and their associated assets within the inventory system, from the nonstationarity of demand distributions as well as from having to use estimates of unknown parameters of assumed probability distributions.

As the size of the budget, B, is decreased, the likelihood that a given legacy of some line item becomes a Bad Buy is increased. Indeed, for cases in which the measure of effectiveness is a monotonically increasing function of the decision variables, it is possible to increase the procurement resources to the point that all Bad Buys would disappear. We would expect the number of occurrences of Bad Buys and the loss

$$L(\underline{b}_p) = z(\underline{x}_p^{**}) - z(\underline{x}_p^{*})$$

to increase as the budget, B_p, becomes more restrictive.

A legacy vector, $\underline{b}_p$, with Bad Buys for a particular objective function, $z_1(\cdot)$, may not contain Bad Buys for another objective function, $z_2(\cdot)$. For example, if our expected sales maximization objective were changed to an objective of maximizing expected customer waiting time saved, a given legacy might no longer represent Bad Buys. Thus high-level policy changes exogenous to the IICS which alter goals of the inventory function of the parent organization may create Bad Buys.

In large-scale inventory systems the line-items are often partitioned into inventory subsystems each having independent LIICS and FICS which may experience different degrees of funding support. Often line items assigned to one subsystem are transferred to the control of another. If

an item migrates from a subsystem enjoying relatively ample funding to a subsystem suffering from tight budget constraints, the legacy for that item will likely contain a disproportionate number of Bad Buys when viewed from the eyes of the recipient.

Since it is generally necessary to estimate demand distribution parameters of assumed distributions using past demand data together with other information that may be available, Bad Buys are often generated by using statistical estimates of the parameters of the demand distributions. Since incurred demands will vary, even with stationary demand distributions (*a fortiori* with non-stationary demand distributions) the estimates based on these changing demands will vary so that the LIICS allocations will attempt to optimize a "moving target" in succeeding periods. Thus even as the occurrence of demands tends to eliminate Bad Buys, the demand process causes changes in the statistically estimated demand parameters which tend to generate Bad Buys through changing evaluations of vectors of stocks.

Bad Buys may be generated through adherence to economic order quantitits developed at an epoch when the state variables, macroscopic (such as funds available) and microscopic (such as demands incurred to date) were less completely known than is the case at the time of allocation. The calculated order quantities may have been optimal with respect to the expected value of the pertinent state variables and

yet not optimal with respect to the realized value of the state variables used in the allocation scheme.

Attrition of Bad Buys is normally accomplished by waiting for demands for the stocks in question. In a commercial context demand may be titilated through advertising or prices may be cut. In a military context consumption is generally not stimulated for the purpose of clearing stocks; thus one might expect the impact of Bad Buys to be greater in the latter context. Disposal actions may be an effective option when essentially all of direct costs are recoupable by advertising in the evening paper a "sale" of items which may be marked down to cost. Even if military salvage actions are an option, if the return on the dollar is on the order of ten cents on the dollar of direct costs and if the funds generated are not quickly made available to the LIICS for reallocation, as may be seen from the situation discussed in Example 3, disposal actions will be undertaken in fewer instances. Thus we expect, and observe, that disposal actions are generally less common and less effective in military inventory systems than in the commercial sectors which enjoy a ready market for the goods stocked.

The maldistribution of resources resulting from Bad Buys obviously reduces the efficiency of an inventory system. Furthermore, because the FICS often considers only the dollar value of current stocks when determining the amount of money to be allocated for the procurement of new stocks, Bad Buys

may also have a critical impact on the budget allocation process. The exact effect of Bad Buys on an inventory objective function has been hard to ascertain. For a given value of assets, A, the rate $\frac{\Delta z}{\Delta B}$ of return available from additional budget increments furnished to the system is greater whenever the inventory is in "bad shape" with regard to $L(\underline{b})$, the measure of the effect of Bad Buys. This assertion follows from the assertion that $L(\underline{b})$ may be increased by starting with an optimal vector $\underline{x}^{**}$ and reallocating the associated asset value to a sequence of progressively worse vectors. In the sequence of vectors $\underline{x}$ thus obtained, the available rates of return measured in terms of changes in $z(\cdot)$ per budget dollar spent are progressively greater. Thus, if the FICS considers only the dollar value of current stocks in making budget allocation decisions, procurement funds may be withheld when Bad Buys and rates of return are high (and funds are likely critically needed) and allocate funds when rates of return are lower.

In this chapter the concept of Bad Buys has been introduced within the context of a budget-constrained multi-item inventory system experiencing random demands. Two examples illustrated the stochastic nature of the development of Bad Buys over time while a third example incorporated the management option of excessing. Operational ramifications of Bad Buys in an IICS have been discussed as have a variety of interacting causes for generation and attrition of Bad Buys in operating inventory systems.

IV. INTERACTIONS BETWEEN THE BUDGET PROCESS AND TYPE-S OPERATING POLICIES

In this chapter, after introductory remarks, Day-One Buys are introduced and defined in the context of a system which attempts to operate using Type-S policies within constraints on financial resources. Several budgeting policies are considered together with Type-S operating policies and their interaction is shown to generate Day-One Buys. It is argued that Type-S operating policies are, in general, incompatible with constrained budgets.

We examine two events which occur when specified budget resource limitations are placed on an administrator who is responsible for the operation of a system with Type-S policies. The first event occurs whenever the random amount of resources required to implement Type-S policies in a period exceeds the available resources. This event gives rise to the "Day-One-Buy Problem" in which an accumulation of unsatisfied demands for resources causes significant and extraordinary resource consumption at the beginning of the succeeding period when resources are again made available.

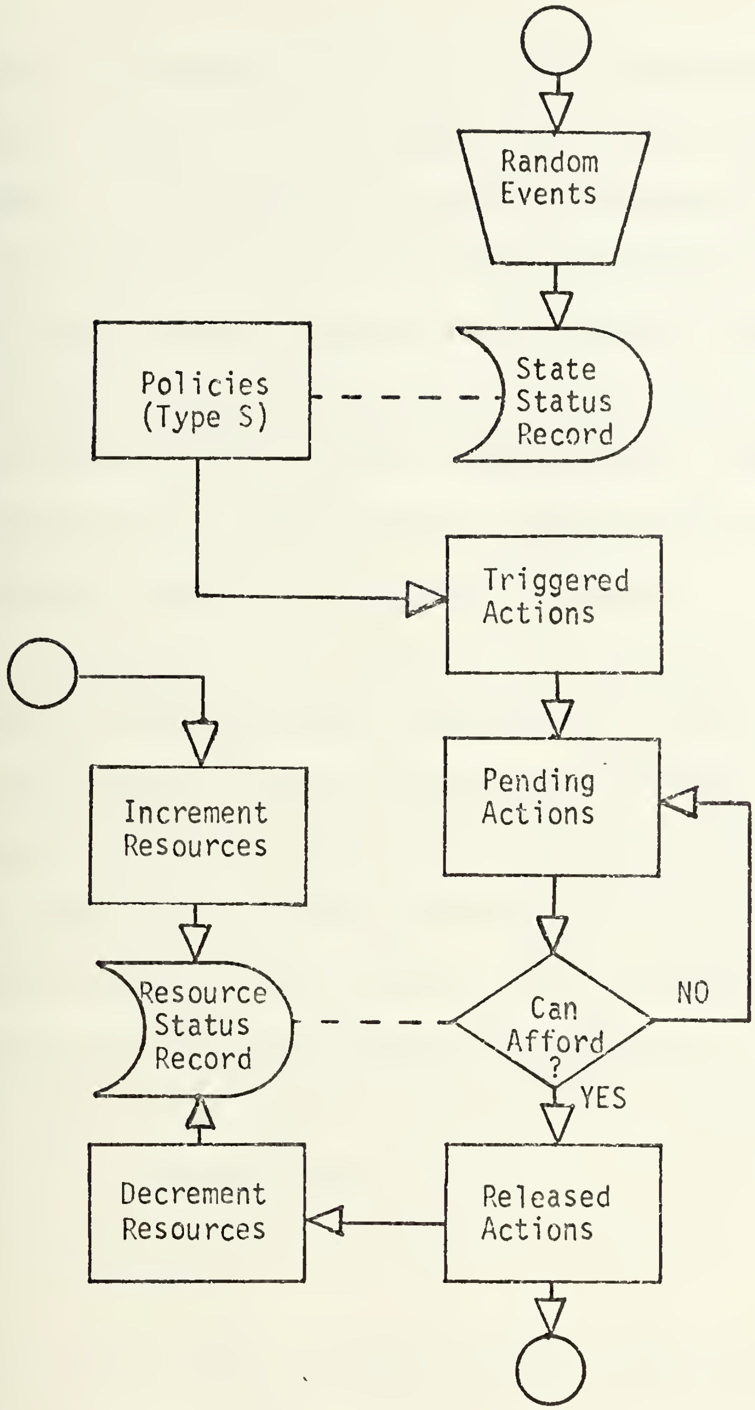
The second event occurs when the budget resources available exceed the resources required to implement the Type-S policy in the period. This event leads to an "End-of-Period Spending Spree" in which an attempt is made to spend all available resources before the next fiscal period.

Although the motivation for a study of the interaction between the budget process and Type-S operating policies has come from the observation of problems experienced in multi-item inventory systems with procurement budget constraints, the underlying problem structure is quite general. Some other examples of Type-S policies are:

1) Fire a fifty-round burst at each suspected guerilla location.
2) Buy 100 shares of AT&T every day that the stock closes below $45.00 per share.
3) Buy five dollars worth of gasoline at every Shell service station encountered.

In each of these cases the Type-S policy fails to adjust for differences between resources available and resources required. Clearly, strict adherence to a Type-S policy may be undesirable or infeasible. In the multi-item inventory problem it is usually necessary to take the stance to make every dollar count.

The operation of a Type-S policy in a system with constrained resources with a "go for broke" adaptation to the constraint is outlined in Figure 4. Random events cause changes in the state status record, e.g., demands cause changes in the vector of inventory positions. The policies operate on information in the state status record and trigger certain actions such as requisitioning of replenishment stocks. The triggered actions are released immediately so long as



OPERATION OF TYPE-S POLICY WITH CONSTRAINED RESOURCES WITH "GO FOR BROKE" ADAPTATION

FIGURE 4

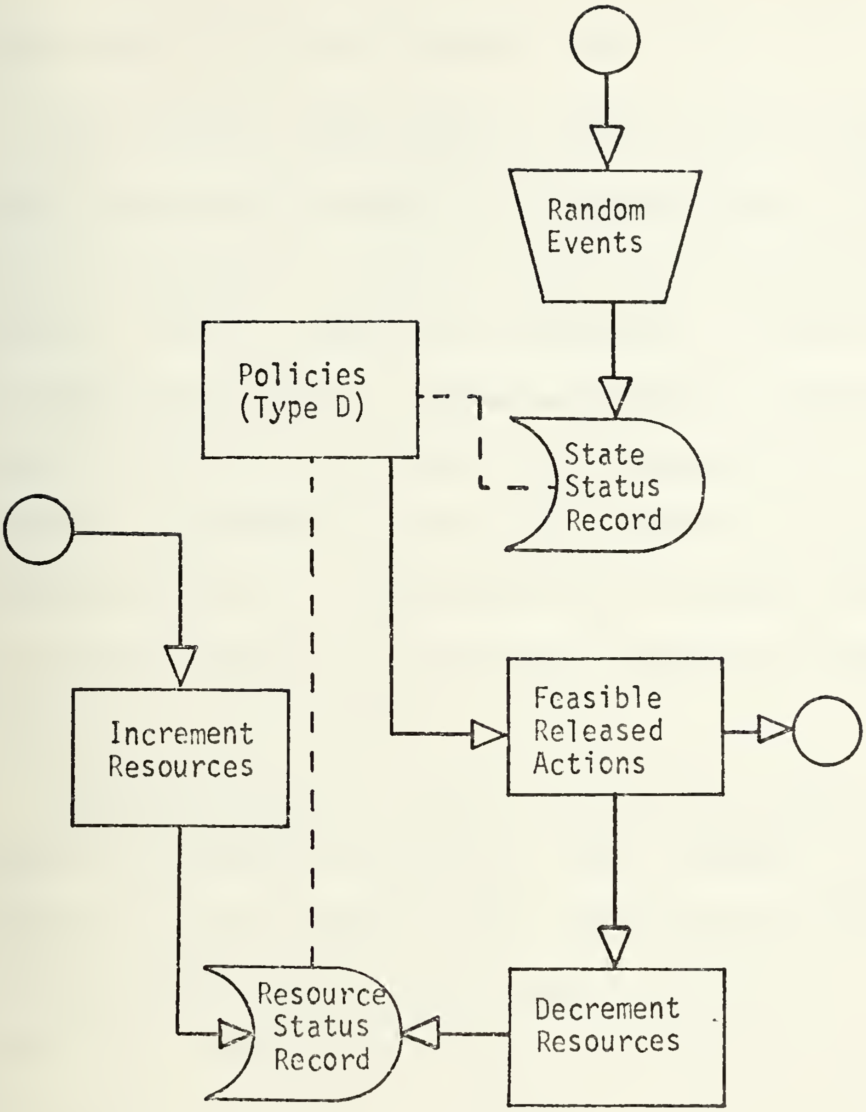
adequate resources are available; otherwise they are held pending receipt of sufficient resources to permit their release. The release of actions consumes the usable resources and periodically more resources are made available. In the context of an Integrated Inventory Control System the Line Item Inventory Control Subsystem might be using (r,Q) policies or (s,S) policies (both Type-S operating policies) constrained by a procurement budget granted quarterly by the Financial Inventory Control Subsystem.

In contrast, the operation of a Type-D policy in a system with constrained resources is illustrated in Figure 5. As before, random events cause changes in the state status record. The policies, however, incorporate information from both the state status record and the resource status record. The actions thus determined may be restricted to those which are feasible with respect to the available resources — and may be optimized with respect to the actual levels of resources available.

A. DAY-ONE BUYS

<u>Definition</u>: Let S_n be the random number of dollars needed to execute actions triggered by Type-S policies in period n and let B_n be the funds then available. Define the budget deficit in period n, D_n, to be

$$D_n = \max \{0, D_{n-1} + S_n - B_n\}.$$



OPERATION OF TYPE D POLICY WITH CONSTRAINED RESOURCES

FIGURE 5

Then period (n+1) has Day-One Buys iff $D_n > 0$.

The obvious counterpart to Day-One Buys are those resources made available to a system during a period which were not needed to execute the Type-S policies during the period. We call these Idle Resources. When implementation of Type-S policies is attempted within a system subject to resource constraints there will generally be simultaneously a positive probability that the resource consumptions triggered within a time period will exceed the budget and a positive probability that the available funds will exceed the needs of the system.

Administrators of LIICS who employ Type-S policies subject to budget constraints must attempt to protect against Day-One Buys and Idle Resources by adjusting the control parameters of their Type-S policies. If reorder points are lowered and order quantities are reduced less procurement funds will be needed. Such actions result, however, in a worsening of customer service effectiveness (more times out of stock, longer delays, etc.), and they require more reorder actions with attendant increase in the processing of transactions. If reorder points are raised and order quantities increased, Idle Resources will likely vanish, but at the cost of an increased likelihood of Day-One Buys. Thus, a conservative administrator who establishes Type-S policies with expected value of resource

consumption far below the available budget will protect against Day-One Buys at the cost of reduced supply effectiveness and high expected Idle Resources. The bolder administrator who attempts to adjust the control parameters so that expected resource utilization exceeds the available budget will achieve a higher level of supply effectiveness at the expense of intolerable Day-One Buys. Without special augmentation from the FICS the effectiveness achieved by the bolder administrator will probably not be sustainable.

If Day-One Buys are not recognized and dealt with formally, they tend to be carried forward without special compensation to the following budget period. This increases the likelihood that the next period will have Day-One Buys. Eventually the replenishment actions of the LIICS may cease while demands incurred tend to creat significant Bad Buys due to the maldistribution of the inventory assets. If Day-One Buys are recognized and dealt with only informally two coexisting systems tend to develop: (1) the "official" Type-S policy system and (2) the "unofficial" policy system which modifies the nominal policy in ways not officially recognized — determining "how we really operate around here."

We discussed in Chapter III the effect that the procurement budget has on the generation of Bad Buys. The more restrictive the budget, the larger is the impact on Bad Buys. Furthermore, we have argued above that the

phenomenon of Day-One Buys is caused by the interaction between the budget process and Type-S operating policies. In the material that follows we look at some commonly used budget allocation procedures, and we show that most lead to potentially critical problems.

B. THE BUDGETING PROCESS RELATED TO DAY-ONE BUYS

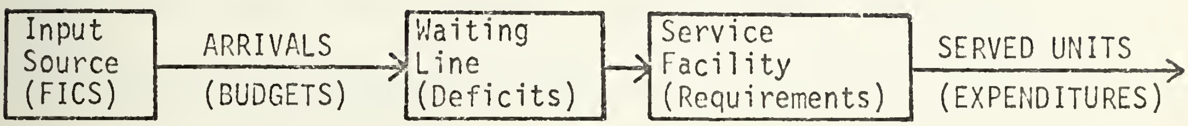
Assume that there are no pending actions at the beginning of period 0. Let S_n be the random number of budget dollars needed in period n to implement a Type-S operating policy. We assume that $\{S_n: n = 1,2, \ldots\}$ constitutes a sequence of independent random variables with distribution function $F(\cdot)$, expected value μ, and variance σ^2. Let B_n be the procurement budget available for ordering stock in period n and let $D_o = 0$. The budget deficit (Day-One Buys) in period n, D_n, is easily seen to be

$$D_n = \max (0, D_{n-1} + S_n - B_n).$$

The budget deficit process $\{D_n: n = 1,2, \ldots\}$ may be viewed as the analogue of the customer waiting time in a single server queue. The budget allocations $\{B_n\}$ play the role of the customer inter-arrival times, and the system needs $\{S_n\}$ correspond to the service time process. Thus, if B_n is random, we have stochastic arrivals and stochastic service times. The idle resources,

$$I_n = \max(0, B_n - S_n - D_{n-1})$$

are analogues of the idle times in the queueing model. In Figure 6 we depict the budget problem in the context of a single-server queue.



BUDGET PROCESS AS A QUEUEING SYSTEM

FIGURE 6

As illustration of the queueing analogy, consider an inventory system with $E(S_n) = \$4000$ for all periods n which allocates a constant amount $B_n = \$4000$ each period. Assume that there is no deficit at the start of period one and that the sequence of values of S_n over the first six periods is as given in Table VIII. The deficits D_n and idle resources I_n are displayed.

Period	S_n	B_n	D_n	I_n
1	\$3500	\$4000	\$ 0	\$500
2	4500	4000	500	0
3	4500	4000	1000	0
4	4000	4000	1000	0
5	3500	4000	500	0
6	4000	4000	500	0

BUDGET DEFICIT - QUEUEING ANALOGY EXAMPLE

TABLE VIII

Although the total budget provided in the six periods is equal to the total resources required, the average value of the deficit is approximately \$583. In the following material we examine various commonly used budgeting policies, and we exploit the queueing analogy to assess the impact of the policies with respect to the expected sizes of budget deficits and the probability distribution of budget deficits.

C. CLASSES OF BUDGET POLICIES

1. $B_n = (1+\alpha)\ E(S_n)$

This budget policy attempts to allocate slightly more or less funds than are needed "on the average." For $\alpha = 0$, the "right" amount of funds are provided "on the average." This is a logical choice of policy if S_n has zero variance since the LIICS makes full use of the resources provided while the FICS allocates no more funds than are actually needed. Unfortunately, this policy has been adopted in large multi-item inventory systems where S_n is random. Experience and simulations alike (see [13], [17]) and well-known analytical queueing results all have shown that the system cannot work under such a combination of budgeting and operating policies with $\alpha \leq 0$. With $\alpha \leq 0$, the analogous queueing situation is the D/G/1 queue with traffic intensity greater than or equal to one. For such a case, the system is guaranteed to "blow up" in the sense that the expected deficits grow without bound as n gets large. If, on the other

hand, α is strictly positive (the traffic intensity is less than one) the stochastic process $\{D_n\}$ possesses a steady-state distribution, but the expected value of D_n may still be large. Marshall [18] has shown that the expected value of the budget deficit process $\{D_n\}$ at steady state is bounded above and below by

$$\frac{\mathrm{Var}\ [S_n]}{2(B - E(S_n))} - \frac{E[S_n]}{2} \leq E(D_n) \leq \frac{\mathrm{Var}\ [S_n]}{2(B - E(S_n))}$$

regardless of the particular distribution of S_n. Table IX gives some numerical examples using the bounds above. Note that the bounds are proportional to the variance of S_n and inversely proportional to the difference between B_n and μ. When S_n is deterministic and constant and $S_n < B_n$ there is no deficit. With even modest variability, however, the steady-state expected value of the budget deficit can be very large. For example, when the standard deviation of S_n is only one-fifth the magnitude of the mean of S_n, the expected budget deficit is between one-half a period's budget and a whole period's budget when $\alpha = 0.02$. It is assumed that this reflects catastrophic failure of the IICS.

In actual practice the true mean μ is unknown. Thus, instead of using the budget policy $B_n = (1+\alpha)\mu$, what is usually done is to replace μ by some statistical estimate. Let $\tilde{S}$ be any unbiased estimator of μ which is independent of S_n. Consider the class of budget policies:

$B_n = (1+\alpha)\mu$, S_n are i.i.d., $\mu = E(S_n) = \$100,000$

Let $c = \sigma/\mu$, where $\sigma^2 = Var(S_n)$

α \ c		0	.2	.4	1.0
.05	U	0	\$ 40,000	\$160,000	\$1,000,000
	L	0	0	110,000	950,000
.02	U	0	100,000	400,000	2,500,000
	L	0	50,000	350,000	2,450,000
.01	U	0	200,000	800,000	7,200,000
	L	0	150,000	750,000	7,150,000

UPPER AND LOWER BOUNDS ON THE STEADY-STATE EXPECTED DEFICIT FOR $B_n = (1+\alpha)\mu$

TABLE IX

2. $B_n = (1+\alpha)\tilde{S}$

With this policy the needs generated in a period are estimated and the budget provided is proportional to that estimate. Because of lead times in the budget process the estimator $\tilde{S}$ may employ only data for periods n-k, n-k-1, ..., where $k \geq 1$. If the estimator perfectly estimates μ the expected deficits are the same as in case 1. above. The sampling variance inherent in the estimator may tend to increase the expected values of both the idle resources and the budget deficit from those experienced in case 1. This class of budget policies includes a large number of cases that may well have been employed in the past. Among these are:

2a) $B_n = (1+\alpha)S_{n-k}$, $k = 1,2, \ldots$

2b) $B_n = (1+\alpha)\overline{S}$ where $\overline{S}$ is a sample mean or a moving average of previous values of $\{S_1, S_2, \ldots, S_{n-1}\}$.

2c) $B_n = (1+\alpha)\hat{S}$ where $\hat{S}$ is a forecast of S_n obtained by using exponential smoothing.

Since D_n and B_n are not necessarily independent for this class of policies (both depend on the values $S_1, \ldots, S_{n-1}$) standard queueing results for GI/G/1 queues are not applicable. Generally speaking, little can be said analytically about the magnitude of waiting times (budget deficits) when the service process and the arrival process are not independent.

In the budget policies discussed above, the FICS does not take into consideration the present plight of the LIICS when determining the budget allocation for the LIICS. Certainly, it would be reasonable for the FICS to "bail out" the LIICS if large deficits exist -- that is, to provide funds for the Day-One Buys in addition to funds for "normal operations during the fiscal period." The following policy considers this sort of feedback.

3. $\underline{B_n = (1+\alpha)\mu + D_{n-1}}$

In this case the FICS funds all deficits and allocates an additional amount equal to some multiple of the expected needs of the system. Now

$$D_j = \max\ (0, D_{j-1} + S_j - B_j) = \max\ (0, S_j - (1+\alpha)\mu)$$

and

$$D_{j-1} = \max\ (0, S_{j-1} - (1+\alpha)\mu)$$

so that the random variables D_j are independent and identically distributed. It is noted that

$$E(D_n) = E(\max(0, S_{j-1} - (1+\alpha)\mu)$$

$$= \int_{(1+\alpha)\mu}^{\infty} (s - (1+\alpha)\mu)\ dF_S(s)$$

For example, suppose the S_j are each normally distributed with mean μ = \$100,000 as before and coefficient of variation $c = \sigma/\mu$. Various values of $E(D_n)$ are given in Table X.

EXPECTED VALUE OF D_n for $B_n = (1+\alpha)\mu + D_{n-1}$, all n

α \ c	0.2	0.4	1.0
.05	\$5728.00	\$13,584.00	\$37,440
.02	7020.00	14,976.00	38,910
.01	7488.00	15,460,00	39,390

TABLE X

As seen in Table X , the expected deficits can be relatively large even when previous deficits are funded each quarter. Similar results can be obtained for the idle resources.

As with policy 1, the parameter μ is unknown and, in actual practice, must be estimated by some statistical estimator $\tilde{S}_n$. This leads to the class of policies:

4. $\underline{B_n = (1+\alpha)\tilde{S}_n + D_{n-1}}$

Assume that $\tilde{S}_n$ is an unbiased estimator of μ $(E(\tilde{S}_n)=\mu)$ and that $\tilde{S}_n$ is a function of the values of the S_j for $j \leq n-1$. Now,

$$E(D_n) = E(\max(0, S_n - (1+\alpha)\tilde{S}_n))$$

where the expectation is taken with respect to the joint distribution of the random variables S_n and $\tilde{S}_n$. By the assumption on $\tilde{S}_n$ and the independence of the S_j this is simply the product of the distributions of S_n and $\tilde{S}_n$. Thus, the expected deficits in period n will depend on the distribution of the estimator $\tilde{S}_n$.

Let S_n be fixed at s and define

$$g(\tilde{S}_n) = \max\ (0, s - (1+\alpha)\tilde{S}_n)\ .$$

Then, $g(\tilde{S}_n)$ is easily seen to be a convex piecewise linear function of the random variable $\tilde{S}_n$. Furthermore, since $\tilde{S}_n$ is unbiased,

$$g(E(\tilde{S}_n)) = \max\ (0, s - (1+\alpha)\mu),$$

and by Jensen's Inequality (see Feller, [19])

$$E(D_n | S_n = s) = E(g(\tilde{S}_n)) \geq g(E(\tilde{S}_n)) = \max\ (0, s - (1+\alpha)\mu) \ .$$

Finally,

$$\begin{aligned} E(D_n) &= E(\max\ (0, S_n - (1+\alpha)\tilde{S}_n) \\ &= E\{E(D_n | S_N)\} \\ &\geq E(\max(0, S_n - (1+\alpha)\mu) \qquad \Box \end{aligned}$$

Thus we have proved

Theorem 1: Let $\{S_j\}$ be i.i.d. random variables with mean μ and let $\tilde{S}_j$ be an unbiased estimator of μ which is independent of all S_k for $k \geq j$. Then the expected deficit for a budget policy $B_n = (1+\alpha)\tilde{S}_n + D_{n-1}$ can be no smaller than

$$\int_{(1+\alpha)\mu}^{\infty} (s - (1+\alpha)\mu)\ dF(s) \ .$$

Intuitively this result is not surprising, for the estimator $\tilde{S}_n$ will have some variability causing extra uncertainty over the case when μ is known.

Clearly, if the objective were to minimize both deficits and idle resources, the optimal budget policy would be a "blank check" policy.

5. $\underline{B_n = S_n}$

This is the financial policy implicitly assumed by most modeling formulations in the literature of inventory control. Here, the budget is random and is equal to the amount of resources required to implement whatever Type-S policies are employed. Such uncertainty in the level of expenses being committed causes financial managers much discomfort. Blank check policies are contrary to the appropriations process of the federal government and are often ruled out by legally imposed budget mechanisms.

In an attempt to limit the risk of too great a budget consumption, financial administrators may adopt a modification of the "blank check" policy which makes available to the LIICS an amount equal to the actual requirements, but only when that amount does not exceed some specified upper bound B.

6. $\underline{B_n = \min\ (S_n, B)}$

This policy completely eliminates Idle Resources, but makes the budget deficit problem severe. No explicit capability is provided for the reduction of any budget deficits which might be generated. For $x \geq 0$, consider that, since $D_n = \max\{0, D_{n-1} + S_n - B_n\}$, then

$$E(D_n | D_{n-1} = x) = x \int_0^{B-x} dF(s) + \int_{B-x}^{\infty} (x+s-B)dF(s)$$

$$= x + \int_{B-x}^{\infty} (s-B)dF(s) > x$$

Eliminating Idle Resources by this policy thus leads in expected value to deficits which increase without bound so long as $P(S > B) > 0$. The reaction to such a trend might lead to its counterpart next discussed.

7. $\underline{B_n = \max (S_n, B)}$

This policy completely eliminates the budget deficit, but at the expense of uncertainty as to the financial assets needed to operate the system until the period has ended. Furthermore, unless B is smaller than the minimum of the essential range of S_n, Idle Resources are not eliminated.

D. IMPLICATIONS OF THE MODELS

Several points seem clear from a quantified study of the implications of variability of demand for constrained resources under Type-S policies.

First, even modest amounts of variability of the amount of resources required to implement Type-S policies from one time period to another within a constrained budget scenario can lead to significant long-term average quantities of resources made available but not used in the periods granted while, for the same sequence of time periods enough pent-up unfunded needs from time to time to significantly decrease system effectiveness. If many locations of a world-wide inventory system operate independently, each with individual procurement budgets, one might expect that some of these activities use less funds than were made available while

others run short of funds and cannot implement the Type-S policies. Where Day-One Buys exist and where Type-S operating policies are designed to generate resource needs in amounts equal, on average, to a fixed periodically granted budget level one would expect Day-One Buys to worsen, statistically, in succeeding periods to the point where extraordinary actions are required to clear the accumulated deficits. In the governmental context such actions may be called reprogramming of funds or supplemental appropriations.

One expects that naval support activities serving widely varying fleet populations, as the fleets are periodically redeployed, would have great difficulty in operating under Type-S inventory replenishment policies. Activities having a relatively stable population of customers and level of activity, reflected in a lesser coefficient of variation of the random variable S_n should be affected in much lesser degree.

Finally, it is noted that the informal adaptations of the nominal, formal Type-S ordering policies as noted in Chapter II are a natural consequence of the adoption of an inventory ordering policy unsuited to the FICS with which it has been coupled. Official retention of unimplementable policies results in parallel systems – one the "official" policy or doctrine and another "the way we do it here." The formal system may thus serve as only an occasionally infeasible point of departure for a LIICS administrator.

Problems of the interaction between the FICS and LIICS have been addressed in the case that the LIICS uses Type-S policies while the FICS provides budgets on a periodic basis which constrain the actions of the LIICS administrator. Either subsystem may have policies which are reasonable (or even optimal) when considered independently within a particular context, yet their interaction can produce profoundly adverse consequences, greatly impairing the performance of the IICS of which they are each a part. In the succeeding chapters we propose an operating system for the LIICS which is designed to operate in an environment with constrained procurement funds. To change the financial control mechanisms to a "blank check" policy as described above would require approval at the highest levels of the executive department and in the congress; conversion of Type-S policies to Type-D policies may be within the capability of a single agency.

V. MODEL AND PROCEDURE FOR LINE-ITEM CONTROL IN A MULTI-ITEM LIICS WITH LIMITED FUNDS

In this chapter we view the exercise of line-item inventory control as a process of transforming resources into new distributions of inventory position over the line items in the inventory. Attributes desired of a LIICS inventory control model which are not generally available in previous models are discussed. The situational context contemplated for use of the model is presented, followed by the model and a solution procedure. The chapter concludes with an example of how the model might be used in a multi-item inventory system with budget constraints.

The essential problems of control in a LIICS with multiple line items are:

1) How much resources to commit at a point in time, and

2) How shall these resources be allocated among the diverse opportunities afforded by the various line items.

In a typical continuous review inventory system, operating with Type-S policies, the first question is answered by a two-stage process. In the first stage, management specifies the reorder point and the order quantity (or the requisition objective) — often on a quarterly basis. In the second stage the application of random demands against stock assets triggers the reorder of a certain quantity of each line item having assets at or below the reorder point. Prior to the

replenishment epoch, there may have been a weighing of stipulated costs in a model which led management to believe that the expected hypothetical costs of this model would be minimized by the choice of order policies specified for the line items. By appeal to the law of large numbers one might hope that these choices, over a large number of budget periods, would be least costly in terms of the hypothesized costs — providing that the stated policies are implemented as designed. While such rationale may be comforting, such Type-S policies are inadequate to assure that, faced with fixed budgets and manpower, it will be possible to implement the policies in a particular period of time. Thus in the world of applications of inventory theory it is not impossible to find inventory policies with claims of optimality which provide infeasible decision responses to random demands in cases where procurement funds are limited.

The determination of the quantity of the available procurement funds to use at a replenishment epoch is not trivial and must be answered with regard to the financial milieu as well as the asset distribution of the LIICS. If funds are more scarce at the procurement epoch than previously anticipated, the line item inventory control system should adapt accordingly. The aggregate financial and workload conditions need to be reflected in the line item allocations made by a multi-item inventory system operating under constraints. In the following chapter we present a model to guide the

LIICS in determining how much of the available procurement funds to use at a particular replenishment epoch.

In a constrained multi-item inventory system, we question whether it is sufficient to consider only those items experiencing demands at the replenishment epoch or whether it may be optimal to order stocks of items at times other than the times they experience demands. The usual varieties of continuous-review (Type-S) systems make this simplification. Actual military inventory systems do not always place orders when the reorder point is hit; further, orders are often placed at times other than at the epochs of demand arrivals. The principal determinant appears to be the relative availability of funds, although sometimes workload considerations affect the determination. Also, batching in the supporting data processing system partially accounts for the non-concurrency of demands and replenishments. Certainly it seems clear that financial and operational (workload) constraints and the relative availabilities of the assets they reflect do change over time so that it may be optimal to place orders for units of items at times other than when they experience demands. Such orders, for example, might be triggered by changes in availability of procurement funds in the constrained inventory system or by changes in estimates of the underlying demand process.

A. ATTRIBUTES NEEDED BY LINE-ITEM ALLOCATION POLICIES

A policy for line item allocation should possess the following attributes:

1. The policy should enable the administrator who is held accountable for the replenishment actions to exercise adaptive control, consistent with policy, in responding to changes in asset distribution, resource availability, system objectives and perceptions of line item attributes.

2. It should consider alternate line item investment opportunities (range and depth) across the inventory at each replenishment epoch.

3. It should provide for determination of stock range questions through an optimization process integral with the replenishment process rather than as a separate, *ad hoc* decision.

4. It should permit implementation and operation without undue expense to either scarce human resources or scarce dollars.

5. It should permit full consideration of the integer nature of replenishment quantities.

6. It should consider the real time inventory position of every line item in determining replenishments for each item.

7. It should provide as an optimization byproduct information useful in determining the adequacy or propriety of the funding levels and the ability to handle the workload;

and provide information useful for examining the balance of procurement budget resources versus operating subsystem resources (i.e., on the relative productivity of additional units of various resources).

8. It should provide a means of assessing the impact on various measures of interest (possible measures of effectiveness) of a particular policy, such as the effect on "requisitions short" of optimizing "sales from stock" and vice versa. It should provide further a means of adjusting decisions to compensate for undesirable consequences of choosing one measure at the expense of others.

The current VOSL LIICS policy lacks most of these attributes whereas the model and procedure presented in Section B satisfy each of the above attributes.

B. THE LINE-ITEM ALLOCATION MODEL AND SOLUTION PROCEDURE

1. Framework

Consider the case in which an administrator, responsible for the replenishment decisions, determines replenishment of stocks of various line items on a periodic basis, say, weekly. Suppose that a fixed amount of procurement budget has been allocated to the replenishment epoch at hand and that a target number of reorder actions has been established as a working constraint for the allocation epoch. The administrator's task is to transform the available resources (procurement funds and operating subsystem assets used in the replenishment process) into replenishment orders for

different line items. By the Principle of Optimality it is known that the present allocation decision, if it is to be optimal, must be optimal with respect to the state of the inventory system at the time of the reallocation decision, regardless of past decisions or states; the state of the inventory system reflects not only the status of individual line items, but also the status of procurement budget accounts, as well as the workload capability of the inventory control operating subsystem.

2. Measures of Effectiveness

The model that we develop permits the choice of several possible objective functions and can be adapted to the case where unsatisfied demands are backordered and to the case where unsatisfied demands are considered to be lost sales. Let $\pi_j > 0$ be the penalty (reward) per unit for item j and let D_j be the demand for item j in a period. Let x_j be the inventory position for item j after ordering in a period. Let $D_j = d_j$. Then the number of sales for item j in the period is given by

$$\begin{cases} d_j & \text{if } d_j \leq x_j \\ x_j & \text{if } d_j > x_j \end{cases}$$

The expected sales for item j is therefore

$$\sum_{d_j=1}^{x_j} d_j P(D_j = d_j) + x_j P(D_j > x_j)$$

which is equivalent to

$$E(D_j) - \sum_{d_j=x_j+1}^{\infty} (d_j - x_j) P(D_j = d_j)$$

We assume that the inventory system seeks to minimize the expected penalty incurred, or, equivalently, to maximize the expected penalty avoided. Mathematically, the objective is to maximize

$$z(\underline{x}) = \sum_{j=1}^{N} \pi_j ((E(D_j) - \sum_{d_j=x_j+1}^{\infty} (d_j - x_j) P(D_j = d_j))$$

Several interpretations and uses of the penalty coefficient π_j are possible. Four are illustrated in Table XI below. Each reflects a formulation of system objective which has been adopted or considered by the Navy Supply System. Alternately, π_j may be taken as a linear combination of various coefficients if the LIICS administrator wishes to weight the individual coefficients.

Reference [20] provides a listing of commonly used objective functions for Navy applications and an indication where each is used.

3. The Line-Item Allocation Model

There are two macroscopic consequences of first magnitude associated with an inventory policy, namely, the

Penalty Coefficient	Objective
$\pi_j = c_j$	Maximize expected sales from stock
$\pi_j = 1/\mu_j$	Maximize the expected requisitions filled as in the VOSL model (μ_j = average quantity of item j demanded per requisition)
$\pi_j = 1$	Maximize the expected number of units issued from stock
$\pi_j = (LT_j\text{+TMNIS-TMISS})$	Maximize expected customer waiting time per unit avoided by issue from stock, where LT_j is the lead time for item j, TMNIS is the calendar time anticipated to process a request which must be referred to the wholesale system and TMISS is the anticipated time to effect issue from stock of a demanded, available item.

INTERPRETATIONS AND USES OF π_j

TABLE XI

investment of money in stocks and the generation of replenishment tasks [5]. It is evident that an actual inventory system with limited resources might be unable to carry out a prescribed inventory policy if either the amount of procurement funds available or the number of replenishment actions which can be processed exceed the available resources. These considerations along with the fact that the x_j must be integers

led to choice of the nonlinear integer programming model:

$$\text{(P3)} \quad \max\ z(\underline{x}) \quad \text{s.t.} \quad \sum_{j=1}^{N} c_j(x_j - b_j) \le B$$

$$\sum_{j=1}^{N} H(x_j - b_j) \le R$$

$$\underline{x} \in X \equiv \{\underline{x}: x_j \in \{b_j, b_{j+1}, b_{j+2}, \ldots\} \text{ for all } j = 1,2, \ldots, N\}$$

where

c_j is the unit price of line item j

x_j is the inventory position of line item j

b_j is the inventory legacy for line item j; brought forward to the replenishment epoch

$H(\cdot)$ is the Heaviside unit function:

$$H(x) = 1, \quad x > 0$$
$$= 0, \text{ otherwise}$$

$\underline{x}$ is a Nx1 vector of inventory positions for the N line items in the inventory

B is the procurement budget available at the reallocation epoch

R is the maximum number of reorder transactions to be considered in the present allocation.

Observe that the model is an integer nonlinear programming problem, separable in the line items. The objective function

$z(\cdot)$ is the sum of $z_j(\cdot)$ which have a discrete domain: $x_j \in \{b_j, b_{j+1}, \ldots\}$. Further, the $z_j(\cdot)$ are non-decreasing functions satisfying a type of concavity on the integers such that, if

$$0 \leq b_j \leq i < k < m,$$

then

$$((k-i)/(m-i))z_j(i) + ((m-k)/(m-i))z_j(m) \leq z_j(k).$$

4. Solution Procedure

Problem (P3) lends itself readily to formulation as a dynamic programming problem, see Bellman and Dreyfus [13]. The recursive functional is given by

$$f_n^*(B,R) = \max_{x_n \in \{b_n, b_{n+1}, b_{n+2}, \ldots\}} \{z_n(x_n) + f_{n-1}^*(B-c_n(x_n-b_n), R-H(x_n-b_n))\}$$

where $f_n^*(B,R)$ denotes the maximal return attainable from consideration of allocation of B procurement dollars and R possible replenishment actions over the first n of the N line items. The dynamic programming solution is thus built up recursively. The dynamic programming formulation does not permit quick and easy solution, however, as the two constraints provide an unpleasant recursion for, say, 10,000 line items.

We selected a more efficient solution procedure that adapts the Generalized Lagrange Multiplier (GLM) method of Everett to a class of inventory problems. Consider the problem (L3) below:

$$\text{(L3)} \quad \max_{\underline{x}} L(\underline{x},\underline{\lambda}) = z(\underline{x}) - \lambda_1((\sum_{j=1}^{N} c_j(x_j - b_j)) - B)$$

$$- \lambda_2((\sum_{j=1}^{N} H\ (x_j - b_j)) - R)$$

$\underline{x} \in X$ and $\lambda_1, \lambda_2 \geqq 0$ with optimal solution $\underline{x}^*(\underline{\lambda})$.

Problem (L3) is the Lagrangian problem associated with (P3). We have that

Theorem 2: If $\underline{x}^*(\underline{\lambda})$ solves the Lagrangian problem (L3), then $\underline{x}^*(\underline{\lambda})$ solves the modified problem (P3) where

$$B = B(\underline{\lambda}) \equiv c_j(x_j^*(\underline{\lambda}) - b_j) \equiv g_1(\underline{\lambda})$$

$$R = R(\underline{\lambda}) \equiv H\ (x_j^*(\underline{\lambda}) - b_j) \equiv g_2(\underline{\lambda})$$

Proof: (see Everett, [14])

If $\underline{\lambda}^{(k)} \geq \underline{0}$ can be found such that the original constraints of (P3) are met with equality through solution of problem (L3), the solution $\underline{x}^*(\underline{\lambda})$ is an optimal solution to (P3). Guidance on how to adjust $\underline{\lambda}^{(k)}$ in the event that $R \neq R(\underline{\lambda})$ or $B \neq B(\underline{\lambda})$ can be obtained from

Theorem 3: Let $\underline{\lambda}^{(1)}$ and $\underline{\lambda}^{(2)}$ be vectors of Lagrange multipliers with $\lambda_k^{(1)} > \lambda_k^{(2)}$, while $\lambda_j^{(1)} = \lambda_j^{(2)}$ for all $j \neq k$. If $x(\underline{\lambda}^{(i)})$ solves (L3) with $\underline{\lambda} = \underline{\lambda}^{(i)}$, for i = 1,2; then

$$g_k(x(\underline{\lambda}^{(2)})) \geq g_k(\underline{x}(\underline{\lambda}^{(1)}))$$

Proof: (see Everett, [14])

In solving problem (P3) through examination of a sequence of solutions to (L3), it is helpful to know a bound on how inferior to the optimal solution an incumbent $\underline{x}^*(\underline{\lambda})$ might be. From consideration of the Minimax Dual Problem we have

Theorem 4: $z(\underline{x}^*) \leq z(\underline{x}^*(\underline{\lambda})) - \lambda_1(B(\underline{\lambda}) - B) - \lambda_2(R(\underline{\lambda}) - R)$, where $\underline{x}^*$ is the optimal solution to P3.

Proof: (see Lasdon, [21])

Following Everett, we separate the N-variable optimization problem into N one-variable problems. Choosing trial values of λ_1 and λ_2 we maximize

$$L_j(x_j,\underline{\lambda}) = z_j(x_j) - \lambda_1 c_j(x_j - b_j) - \lambda_2 H(x_j - b_j)$$

over the set of values permitted, namely $x_j = b_j, b_{j+1}, \ldots$. We consider two cases — that of ordering some positive quantity of line-item j and that of not ordering item j. In the first case, the separated Lagrangian expression is

maximized utilizing

Theorem 5: Let B_j^+ be the set $\{b_j+1, b_j+2, \ldots\}$. Then $L_j(x_j,\underline{\lambda})$ is maximized over $x_j \in B_j^+$ at the smallest value of $x_j \in B_j^+$ which satisfies the inequality

$$\pi_j P(D_j > x_j) \leq \lambda_1 C_j.$$

Proof: Let $x_j^{\#}$ be the smallest value of $k_j \in B_j^+$ which maximizes $L(x_j,\underline{\lambda})$. Being an integer, it is necessary that $L_j(x_j^{\#}+1,\underline{\lambda}) - L_j(x_j^{\#},\underline{\lambda}) \leq 0$ and $L_j(x_j^{\#},\underline{\lambda}) - L_j(x_j^{\#}-1,\underline{\lambda}) > 0$. Thus $x_j^{\#}$ is the smallest value of $x_j \in B_j^+$ such that

$$\Delta L_j(x_j,\underline{\lambda}) = L_j(x_j+1,\underline{\lambda}) - L_j(x_j,\underline{\lambda}) \leq 0$$

Since $\Delta L_j(x_j+1;\underline{\lambda}) = \pi_j P(D_j \geq x_j+1) - \lambda_1 C_j$ we have that $x_j^{\#}$ is the smallest $x_j \in B_j^+$ such that $\pi_j P(D_j \geq x_j^{\#}+1) \leq \lambda_1 C_j$.$\square$

Denote the maximizing value of case one by $x_j^{\#}$. Next, evaluate $L_j(b_j,\underline{\lambda}) = z_j(b_j)$, considering the second case. Since $x_j \in \{b_j\} \cup B_j^+$ we have the maximizing value $x_j^*(\underline{\lambda})$ from

Theorem 6: $x_j^*(\underline{\lambda}) = b_j$ if $L_j(b_j,\underline{\lambda}) \geq L_j(x_j^{\#}(\underline{\lambda}),\underline{\lambda})$
$= x_j^{\#}(\underline{\lambda})$ otherwise.

The vector $\underline{x}^*(\underline{\lambda})$ thus obtained, together with $z(\underline{x}^*(\underline{\lambda}))$, the number of reorders and their cost to the

procurement budget are determined. An upper bound $U(\underline{\lambda})$ is computed from each such iteration on a pair of Lagrangian multipliers using Theorem 4:

$$U(\underline{\lambda}) \equiv z(x^*(\underline{\lambda})) - \lambda_1(B(\underline{\lambda}) - B) - \lambda_2(R(\underline{\lambda}) - R) .$$

A decision is then made whether to choose another pair of Lagrangian multiplier values or whether the solution thus obtained is close enough. Decreasing the non-negative multiplier value tends to use more of the corresponding resource, increasing it uses less. When the replenishment actions generated by a point (λ_1, λ_2) in the non-negative quadrant of E^2 exactly consume the available resources, B and R, we know that the solution is optimal. Due to the integer nature of the problem exact equality may be impossible because of "duality gaps." In such a case the investment of a great deal more effort through a dynamic programming formulation and solution procedure might obtain a marginally superior solution to (P3). Alternatively, and much more efficiently, it may be possible to compare two vectors $\underline{x}^*(\underline{\lambda}^{(1)})$ and $\underline{x}^*(\underline{\lambda}^{(2)})$, one feasible and near-optimal and the other "super-optimal" (i.e., with greater value of objective function than that achievable with budget B and number of reorders R) and slightly infeasible, line item by line item and "fill the duality gap" on an item-by-item basis, incrementing values from one vector to the other until

there is sufficiently small slack in the constraints. In many instances this latter procedure will not be necessary.

The solution procedure results in an (r(t), R(t)) inventory policy, interpreted as a policy in which the reorder point and requisitioning objective vary as dynamic functions of the distribution of assets in the inventory, the presence of bad buys, the workload implications, and the availability of funds. Perhaps of primary importance, the procedure provides the accountable LIICS administrator with an instrument under his control with which he can live within budget or other constraints, respond to a number of changes in objective function, and obtain the "shadow price" of the procurement budget or replenishment action resources.

Experience in implementing the solution procedure through an interactive computer program furnished in Appendix A has pointed up the great sensitivity of the resource implications of small differences in the multiplier values when some line items with large unit prices are at the margin; relatively large overexpenditures of constrained resources are often accompanied by relatively small increments in effectiveness measures. This latter observation complements the often quoted result of classical inventory theory that the total cost function, as a function of the order quantity, is relatively flat near the optimum. A traditional (s,S) inventory policy (Type-S) results from

the operating policy reflected by choosing just one value of the Lagrangian multiplier pair (λ_1, λ_2) and making just one pass through the inventory, ordering the resulting number of line items in the indicated quantities.

C. ILLUSTRATIVE EXAMPLE

Using a computerized program such as that furnished in Appendix A, the accountable inventory controller, operating in the Line Item Inventory Control Subsystem, would sit down at a remote computer terminal, having an idea as to how much of his available budget he was willing to spend, and how many reorders were tolerable. Further, he would know whether these figures were firm constraints or rough guidelines. After signing in to the computer system he need only initiate execution of the computer program and input data such as unit costs, inventory position, lead time, and demand parameters. Certain program constants, such as the shadow prices of procurement budget and workload constraint, budget available, constants selecting the objective function for optimization and possibly other objective functions for evaluation might be entered unless the accountable person elects to use default values stored in the computer files. A message is returned to the terminal by the computer that the values of program constants provided have generated XXXX stock replenishments costing $XXXXXX.XX and achieving objective function values XXXXXX.XX which compare to a maximum attainable (for the

given budget of $XXXXXXX.XX and XXXXXXX reorders) of no more than XXXXXXX.XX, whereas if infinite resources were available the objective function value might be as great as XXXXXXXXXX.XX. Several other objective function values attained by the present candidate decision vector (such as expected sales, expected units issued from stock, etc.) might also be printed. The accountable person reviews information obtained thus far and determines whether to execute the present candidate decision (generating automated replenishments), whether to print on hard copy the candidate decision (for manual review), or whether to iterate the computation. The computer program queries the accountable person whether the same objective function is to be used and whether the same shadow prices are to be used. Any required changes are typed in at the console and the computer shortly types a message describing attributes of interest for the current candidate decision vector. Alternate decision vectors can be generated at the console or read in through other terminals and the objective function values attained by these vectors, together with their costs and number of transactions required may be computed. The above procedure can be performed for all line items of an inventory or for any identifiable subset of these items.

In changing the shadow prices which control the GLM optimization process, increasing the value of a Lagrangian multiplier will cause less consumption of that resource.

Sensitivity analysis for the two resources is readily available. Regardless of the choice of objective function used in the optimization, the value of each of the four "pure" objective functions attained by $\underline{x}^*(\lambda)$ is displayed. Together with information on the bounds, the user is presented quantified informatiin on trade-offs in one measure of effectiveness obtained at the expense of decrease in other measures. One can thus determine, for example, the percentage loss in attainable expected sales resulting in a decision to optimize the customer-time-saved measure.

A model, solution procedure and framework for use has been proposed for use under the assumption that the LIICS administrator knows how much resources are to be allocated and whether these resource quantities are goals or constraints. In the following chapter a model and solution is presented to guide the decision of the LIICS administrator in determining a resource consumption plan. It is intended that the models in this chapter and the subsequent chapter would interact in that information obtained in the one would be used in the other as the LIICS operates in real time.

VI. DEVELOPMENT OF A LIICS BUDGET ALLOCATION PLAN

In the preceding chapter it was assumed that the LIICS administrator knew at each allocation epoch how much procurement funds were then available; further it was assumed that he knew whether the amount intended for allocation was a constraint or a goal with some variability permitted. In this chapter a procedure for determining a budget allocation plan is presented. The line-item allocations at an epoch are viewed as a one-input, one-output production process. In this process, the resource consists of procurement funds and the product is the improvement in value of objective function attained (over the value of the legacy inherited at the epoch). The nature of this production function is discussed. Means of estimating this production function through exploitation of properties of the GLM solution procedure of the line-item allocation model of the previous chapter are presented. Solution of the budget allocation model used in preparation of the budget allocation plan is described, followed by examples demonstrating preparation of a budget allocation plan with associated effectiveness attainment plan and projection of shadow prices to be used in the line-item allocation plan at each allocation epoch covered by the plan.

By budget allocation plan is meant a vector of dimension, E, equal to the number of allocation epochs remaining in the

period. The elements of the vector, $\underline{B}_p$, are B_p, B_{p+1}, ..., B_{p+E-1}. Each element has as its value the planned allocation of funds at the corresponding allocation epoch.

We suppose that the objective of the budget allocation model is to maximize the expected value of

$$\sum_{K=p}^{p+E-1} z_K(\underline{x}_K^*) ,$$

measured at allocation epochs after allocation. We construct a simple, deterministic model for this purpose. In essence, we use properties of the GLM optimization procedure used in the line-item allocation model as we solve another non-linear optimization problem using the GLM method. The answers to the latter problem make it possible for us to estimate the value of the triple $(B_p, \lambda_p^*, z(\underline{x}(\lambda_p^*)))$ at each allocation epoch remaining in the fiscal period.

By $z(B_p)$ is meant the optimal value of the line-item allocation model objective function at epoch t_p, viewed as a function of the available procurement funds, B_p. It is necessary for our purposes to make some assumptions about the form of the production function $z(B_p)$. We assume all other relevant factors do not change significantly from one allocation epoch to another within the fiscal period. By $z(B_p(\lambda^{(i)}))$ is meant the optimal value of the line-item allocation model objective function at epoch t_p generated by a choice $\lambda^{(i)}$ of shadow price (Lagrangian multiplier) for

the procurement budget, using a GLM solution procedure as outlined in the preceding chapter. Since there are in reality two constraints in the line-item allocation model, the quality of the estimates generated through the process of this chapter depend on the workload shadow price assuming a role of secondary importance in the optimization process. We thus consider properties of problem (P1) as proxy for those properties of (P3).

A. SHAPE OF THE PRODUCTION FUNCTION

Examination of the form of the class of objective functions considered reveals that they are bounded above by

$$z_{max} = \sum_{j=1}^{N} \pi_j \sum_{i=0}^{\infty} i[P(D_j = i)] = \sum_{j=1}^{N} \pi_j E(D_j)$$

and further that they are bounded below by zero where $\pi_j \geq 0 \ \forall_j$. When a GLM procedure is used for optimization, we obtain a set of triples $(B(\lambda_p^{(i)}), \lambda_p^{(i)}, z(\underline{x}(\lambda_p^{(i)})))$ associated with each trial value $\lambda_p^{(i)}$. From GLM theory we know that if $\lambda^{(1)} < \lambda^{(2)} < \lambda^{(3)}$, then the budget expenditure resulting from each multiplier will <u>decrease</u> monotonically (or not change), as will the value of the measure of effectiveness evaluated at the point $\underline{x}^*(\underline{\lambda})$ attained by each GLM procedure trial solution. Further we know that points $(B(\lambda^{(i)}), z(B(\lambda^{(i)})))$ generated by the $\{\lambda^{(i)}\}$ can be

used to construct a convex polyhedron containing the true curve $z(B_p)$ through application of the bounds cited above, as in Figure (8). To the extent that it is true that the demands experienced by the inventory system between allocation epochs generates approximately the same function $z(B_p)$ for succeeding values of $p = 1,2,3, \ldots,$ then it is reasonable to use a relatively simple, deterministic model to solve the budget allocation problem; we drop the subscript p.

B. PROPERTIES OF B(λ)

We consider properties of problem (P1) as proxy for problem (P3); they hold for problem (P3) where the workload constraint is not binding.

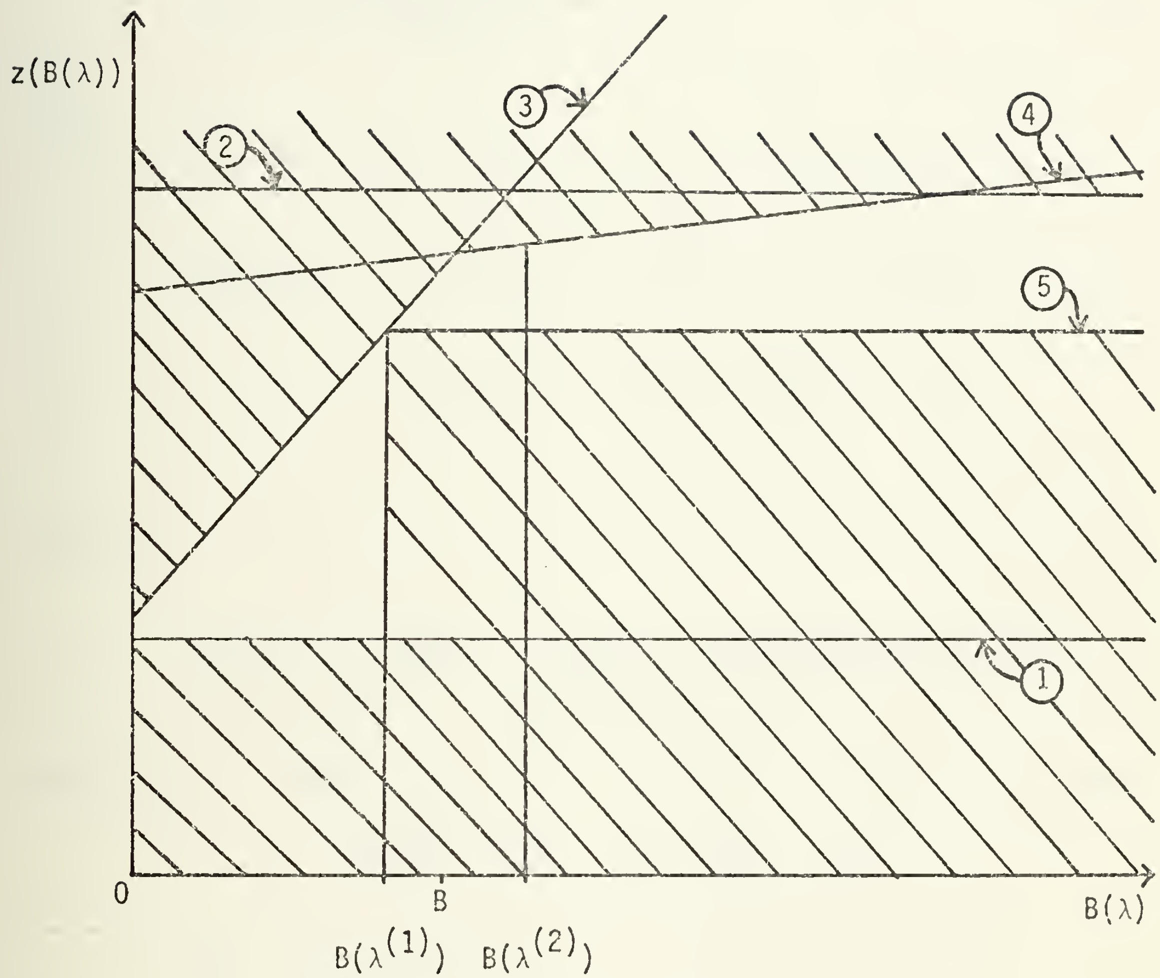
Theorem 7: $B(\lambda)$ is a monotone non-increasing step function of $\lambda \geq 0$.

Proof: $B(\lambda) = \sum_{j=1}^{N} c_j(x_j^*(\lambda) - b_j)$, where the x_j are restricted to the integers so that $B(\lambda)$ is a step function. Everett has shown that the amount of a resource consumed is a non-increasing function of its multiplier, where consumption of any other resources does not change.

Theorem 8: $\lim_{\lambda \to +\infty} B(\lambda) = 0$

Proof: By Theorem 5 we have that

$$x_j^{\#}(\lambda) = b_j + 1 \quad \text{if } \lambda \geq \pi_j P(D_j > b_j+1)/c_j \ .$$



BOUNDS ON z(B)
FIGURE 8

Legend:

1 is a lower bound from $B(\lambda=\infty)(z(\underline{b}))$

2 is an upper bound from $B(\lambda=0)(z_{max})$

3 is an upper bound from $B(\lambda^{(1)}) < B$

4 is an upper bound from $B(\lambda^{(2)}) > B$

5 is a lower bound from $B(\lambda^{(1)}) < B$

z(B) is known to lie wholly between the shaded areas.

Further, whenever

$$\lambda \geq \frac{\pi_j P(D_j > (b_j+1))}{c_j} ,$$

then

$$L_j(b_j,\lambda) = z_j(b_j) \geq L_j(b_j+1,\lambda) = z_j(b_j+1) - c_j\lambda$$

$$= z_j(b_j) + \pi_j P(D_j > (b_j+1)) - c_j\lambda$$

so that

$$B(\lambda) = 0 \quad \text{for } \lambda \geq \max_{j=1,N} \{\pi_j P(D_j > b_j+1)/c_j\}. \square$$

Theorem 9: If $P(D_j > i) > 0$ for all $i = 1,2, \ldots$ and

$x_j^*(\lambda) > b_j$ for some $\lambda > 0$

then $\lim_{\lambda \to 0^+} B(\lambda) = +\infty$

Proof: For some j meeting the hypothesized conditions, $x_j^{\#}(\lambda)$ increases without bound as $\lambda \to 0^+$, by Theorem 5. $\square$

Theorem 10: $\{B(\lambda)\}$ generated by values of λ constitutes a discrete set.

Proof: The decision variables $\underline{x}$ are restricted to a discrete set; clearly their procurement cost $\sum_{j=1}^{N} c_j(x_j - b_j)$ can take on only discrete values. $\square$

C. PROPERTIES OF $z(B(\lambda))$

We continue to use problem (P1) as proxy for (P3) and have

Theorem 11: $z(B(\lambda))$ is a monotone strictly increasing function of $B(\lambda)$.

Proof: $z(B(\lambda))$ is assured to be a monotone non-decreasing function of $B(\lambda)$ since the feasible regions for succeedingly larger values $B(\lambda)$ include those of smaller values $B(\lambda)$. Strict monotonicity is assured by requiring, as indicated in Theorem 6, that

$$L_j(x_j^{\#}(\lambda),\lambda) > L_j(b_j,\lambda) \quad \text{for} \quad x_j^*(\lambda) > b_j. \ \square$$

Theorem 12: $z(B(\lambda))$ is a concave function of $B(\lambda)$ for all points in its domain: $\{B(\lambda)$ generated by values of $\lambda \in [0,+\infty)\}$.

Proof: Follows from the Lambda Theorem of Everett [12]. $\square$

While using the GLM optimization procedure, values of λ are chosen, $\underline{x}^*(\underline{b},\lambda)$ are computed as described above, together

with $B(\lambda)$ and $z(\underline{x}^*(\underline{b},\lambda))$. Thus points may be obtained relating $z(B(\lambda))$ the "product" to $B(\lambda)$ the "resource." From consideration of Theorems 3 and 4 it is clear that the slope of line segments adjoining each point thus obtained must decrease as points in product-resource space having successively greater values of resource $B(\lambda)$ are considered. Further we know that

Theorem 13: Let $B(\lambda^{(i)}) < B < B(\lambda^{(j)})$. Then

(i) $z(B) \geq z(B(\lambda^{(i)}))$ by restriction

(ii) $z(B) \leq z(B(\lambda^{(j)}))$ by relaxation

(iii) $z(B) = z(B(\lambda^{(i)})) + \lambda^{(i)}(B-B(\lambda^{(i)}))$ restated from Theorem 4,

while, following directly from the objective function we have

Theorem 14: $\lim_{B\to\infty} z(B) \equiv z_{max} = \sum_{j=1}^{N} \pi_j E(D_j)$

A familiar exponential function may be used as rough approximation to the production function in the case of no Bad Buys; e.g., in solving (P2). Approximately,

$$z(B) = z_{max}(1 - \exp(-\nu B)$$

where z_{max} is given in Theorem 14 and ν is a parameter to be estimated. $z(B)$ is readily seen to be a continuous concave function with $z(0) = 0$ and with $\lim_{B\to+\infty} z(B) = z_{max}$ as required

by Theorem 14. Further, its slope decreases monotonically, consistent with Theorem 12. Where there are Bad Buys, then the rate of increase in objective function for marginal dollars invested, $\Delta z(B)/\Delta B$, will tend to be greater for the total assets invested, since the maldistribution of stock assets means there are relatively many "good new investment opportunities."

At each allocation epoch the production function tends to yield diminishing marginal returns, while demands incurred between replenishment epochs tend to make available greater rates of return $\Delta z(B)/\Delta B$. Thus our model leads to a policy of smoothing the amounts of budget to be allocated at successive line-item allocation epochs.

When "too much" is spent early in the fiscal period, the relatively small increment to $z(\underline{x})$ obtained by the "last dollars spent" at the early epochs does not compensate for the subsequent inability to take advantage of relatively great rates of return available in the later epochs. In the latter epochs the Bad Buy loss function $L(\underline{b}_p)$ would be relatively great.

D. BUDGET ALLOCATION MODEL

It may be that we do not wish to place equal value on the measure of effectiveness at allocation epochs throughout the quarter. We may have a time preference so that we wish to discount the return at each successive epoch. For example, a LIICS may be penalized in future budget allocations

only for Bad Buys existing at the end of a fiscal period, although it is rewarded for issues throughout the period. In a retail system one might prefer to spend more early in the period, providing a slightly greater level of protection while there remains time for some attrition of any Bad Buys. In this case one might choose the model

$$\text{(P4)} \qquad \max_{\underline{B}} \sum_{k=p}^{p+E-1} c_k W(B_k); \qquad \underline{B}, \text{ an E-vector}$$

$$\text{s.t.} \quad \sum_{k=p}^{p+E-1} B_k \leq B$$

where $\{c_k\}$, for $k = p, p+1, \ldots, p+E-1$, are a set of constants, and $z(B)$ is approximated by a non-decreasing concave function $W(B)$ using information obtained in the solution to the LIICS allocation model. Forming an associated Lagrangian optimization problem (L4), associated with (P4), we have

$$\text{(L4)} \qquad \max_{\underline{B}} L(\underline{B},\mu) = \sum_{k=p}^{p+E-1} c_k W(B_k) - \mu\left(\sum_{k=p}^{p+E-1} B_k - B\right)$$

for $B_k \geq 0$ for all k and $\sum_{k=p}^{p+E-1} B_k \leq B$, where B is the amount of budget remaining available at time p. Examination of the Kuhn-Tucker conditions [21], necessary and sufficient for determining the optimal solution for this concave programming problem, yields the following result:

$$c_k \frac{dW(B_k)}{dB_k} = \mu, \quad \text{for all } k = p, p+1, \ldots , p+E-1.$$

Since $W(B_k)$ is constructed to be monotone non-decreasing and concave, we know that the constraint will be binding, yielding additional information that $\sum_{k=p}^{p+E-1} B_k = B$. Since we anticipate solving the LIICS allocation by a GLM procedure, we have

$$\frac{dW(B_k)}{dB_k} \tilde{=} \lambda_k^*,$$

as the shadow price, λ_k^*, is approximately equal to the rate of change in the objective function per increment of the resource, so that the sequence of optimal Lagrangian multipliers throughout the period in this simple model would be governed by the relationship

$$\lambda_k^* \tilde{=} \frac{\mu}{c_k}$$

and μ would be determined by the requirement that the budget B would be just exhausted at the final allocation. For example, where $c_k = .99^{k-1}$, corresponding to about a 1 per cent discoutn per allocation period, and where $\mu = .8$, we have

k:	1	2	3	4	5	6	7	8
λ_k^*:	.8	.808	.816	.824	.833	.841	.850	.858

where λ_k^* is an estimate of the final budget constraint multiplier to be used in the LIICS allocation solution procedure at epoch k. This succession of values would correspond generally to successive values of $B_k(\lambda_k^*)$ which decrease as λ_k increases. Suppose a discounting factor B, and the number of allocations remaining in the fiscal period, E, are given and that $z_p(B_p(\lambda))$ in successive epochs is essentially the same concave and monotone increasing function of $B(\lambda)$. Then it is possible to use the GLM procedure to determine a budget allocation plan for the remainder of the period. The plan can be stated in terms of estimated final Lagrangian multiplier values for (P1), λ_p^*; the master (budget allocation mode) Lagrangian multiplier, μ; the estimated budget to be spent at time p, B_p; and the projected effectiveness value, $z(\underline{x}_p^*)$, for each remaining allocation epoch in the fiscal period.

E. BUDGET ALLOCATION PLAN EXAMPLE

As a simple example, suppose $z_{max} = 1$ and consider $z(B_p) = 1 - \exp(-B_p)$ while $B = .9$ and where $\sum_{p=1}^{3} B_p = B = 3$. Applying our optimality condition we have

$$\exp(-B_3) = \mu, \qquad \exp(-B_2) = \frac{\mu}{.9}, \qquad \exp(-B_1 = \frac{\mu}{(.9)^2}$$

We need to determine μ so that these relationships hold while the budget $B = 3$ is just consumed.

To obtain a solution to this problem, determining B_1, B_2, B_3 and μ, we use a GLM procedure. Our starting value $\mu^{(0)}$ is obtained by consideration of the case where the discount factor B is equal to one. Since

$$\frac{dz(B_k)}{dB_k} = \mu$$

for all periods k, thus $B_i = B_k$ for $i \neq k$, thus $B_i = \frac{1}{E}(B)$. Here $B_i = \frac{B}{N} = \frac{3}{3} = 1$, so that $\exp(-1) = \mu^{(0)} = .36788$. Solving for B_i yields

i:	1	2	3
B_i:	1.23	1.11	1

We see that the total of the B_i is more than the amount available, so that our Lagrangian multiplier value, $\mu^{(0)}$, was too small. Now let $\mu^{(1)} = .408$; then we obtain

i:	1	2	3
B_i:	1.107	.996	.897

in which case the sum of the B_i is within one-half of one per cent of B, which we may consider close enough. Using this value of the Lagrangian multiplier for the budget

allocation model we obtain the following allocation plan for the three remaining allocation epochs of this fiscal period:

<u>Now</u>	<u>Next Time</u>	<u>Final Time</u>
Spend 1.107	Spend .996	Spend .897
use λ = .408	use λ = .453	use λ = .503
attain $z(\underline{x})$=.670	attain $z(\underline{x})$=.631	attain $z(x)$=.593

In this chapter a model has been presented which complements and interacts with the model presented in the previous chapter. In conditions where the form of the optimal return function of the line-item allocation model, viewed as a function of the budget available for expenditure at the epoch, does not significantly change within the fiscal period, the budget allocation model presented may be useful as a guide as to how much available resources to allocate at a given epoch. Further, the model just presented may be used to project the effectiveness to be attained and to estimate the value of procurement budget Lagrange multiplier to be used at the each allocation epoch remaining in the fiscal period.

If it is considered that the production function $z(B)$ will change in scale, although not radically in form, it would be possible to suitably modify the above procedure. If it develops that the production function $z(B)$ changes

markedly in form and scale from allocation epoch to allocation epoch it may be preferable to estimate it as a function of variables external to the models of the Integrated Inventory Control System, rather than in modeling efforts which presuppose the variety of system stability assumed above.

VII. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

A. CONCLUSIONS

It is concluded that the adoption of Type-S inventory policies in Line Item Inventory Control Subsystems of Integrated Inventory Control Systems which operate with fixed procurement budgets and fixed transaction-processing resources is undesirable where there is even rather modest variability in the amount of such resources that are required from period to period to operate under the policies. Further, it is to be expected that, in such cases, the lack of ability to exercise fine control of the inventory through the execution of approved policy will tend to cause the development of formal systems operating in parallel with *ad hoc*, individualistic, and non-optimal modifications of the formal systems.

It is concluded that the presence of Bad Buys in Inventory Systems can significantly reduce the attainable effectiveness of the inventory system for the given value of stock assets; Bad Buys are a normal consequence of operating inventory systems under conditions of uncertainty and failure to account for their influence can lead to significant bias in estimates of effectiveness attainable and consistent failure to meet designed effectiveness in multi-item inventories operating under constraints.

It is concluded that a feasible, efficient method of effecting line item Inventory control is available using an adaptation of Everett's Generalized Lagrangian Multiplier Method and interactive computer programming, which gives the accountable person a valuable tool in doing the best possible with the assets actually available at the time of decision. Further, that the use of a GLM procedure provides valuable information for financial managers as to the relative effectiveness of additional procurement funds versus additional transaction processing capability. If more than one LIICS were linked financially through control of the procurement budget by a common FICS, information on the relative contribution of inventory procurement budget dollars toward specified objectives might serve to sharpen financial control of this asset, contributing toward greater system effectiveness.

It is concluded that the Day-One-Buy Problem is amenable to operations research analysis, and is a subject of significance to the many levels of the governmental hierarchy. Failure to accommodate satisfactorily the interaction of Type-S policies and the policies governing the provision of resources necessary to execute the policies leads concurrently to idle resources, resource deficits and ad hoc policy modification on the informal level, reflecting loss of effective policy control by management.

In April 1974 Secretary Clements initiated action within the Office of the Secretary of Defense which has resulted in the current project RIMSTOP which has in its charter, among other items, responsibility to develop standard stockage policies (range and depth) for all of the Department of Defense "below the wholesale level" – which would include the retail supply systems of the various services. It is argued that such efforts are not likely to result in significant improvements, and may be counterproductive, if such policies are developed without regard for their ultimate feasibility of implementation within the organizational structure and attendant compartmentalized constraints on resources. An example would be the adoption of Type-S policies in a system with budget constraints.

B. SUGGESTIONS FOR FURTHER RESEARCH

The author is aware of no quantified analysis of the influence of Bad Buys as defined above on the operation of inventory systems. Using the computer program developed herein one could evaluate sample legacy vectors obtained at random from various inventory systems, evaluate the assets, reoptimize starting from zero legacy subject to the constraint that the new inventory position could have value no greater than the previous legacy. The difference in effectiveness measure attained is one estimate of the difference in system effectiveness attainment lost due to Bad Buys.

Statistical estimators of demand parameters could be evaluated by simulations of actual demands being processed by different estimators, resulting in different allocation decisions for a given inventory policy and different attained levels of Bad Buys.

Having defined the Day-One-Buy Problem and shown that this problem, of very general occurrence, falls into one or another categories of queueing models, the door has been just opened to this important area in Chapter IV. It is expected that some types of Budget Policies found in existence, when coupled with a Type-S policy, will be shown to have unfavorable consequences – such as being dominated by other policies in both consideration of Budget Deficit and Idle Resources. The effect of time lag in compensating for deficits or trends in such systems remains an open subject. The effect on the larger system supported by the subsystem controlling the inventories with a Type-S policy – or expending resources in other pursuits in accordance with a Type-S policy remains unexplored. Opportunities to shift some of the inherent variability from variability of performance levels under strict budgets toward variability of resource consumption under specified performance standards need to be sought out and evaluated.

APPENDIX A: COMPUTER PROGRAM

```
      DIMENSION KDEM(5,100),TMLD(100),IDUES(100),IONHD(100),
     1PRICE(100),SZRQN(100),INVNEW(100),PI(100),ZWT(100),TMSVD(
     2100),QUAL(100),KFREQ(5,100)
      COMMON DEMFCT(100), DEVFCT(100),IPOS(100)
      LOGICAL QUAL
      N = 100
      NORDER = 0
      CSTBUD = 0.
      DATA  IC1/'YES'/,IC2/ 'NO'/
      FLAG = 0.
      WRITE(5,137)
  137 FORMAT('  READ IN INITIAL VALUES.')
      READ(3,10) N,(ZWT(I),I=1,4),SHADB,SHADW,TMISS,TMNIS,U,R,WRKLD,
     1BUDGET
      DO 731 L=1,N
  731 READ(4,20) (KDEM(I,L),I=1,5),DEMFCT(L),PRICE(L),(KFREQ(I,L),I=1,5)
     1,DEVFCT(L),TMLD(L),IDUES(L),IONHD(L)
10    FORMAT(I4, 8F5.2,3F6.2,F10.2)
20    FORMAT(5I3,1X,2F5.2,1X,5I2,F6.2,5X,F2.1,7X,I2,I3)
21    FORMAT(5I3,1X,2F6.2,1X,5I2,F7.2,5X,F3.1,7X,I2,I3)
      DO 22 J=1,N
      DEM= 0.
      DNUM=0.
      DO 23 K=1,5
      DDEM=KDEM(K,J)
      DFREQ=KFREQ(K,J)
      DNUM=DNUM+DDEM
   23 DEM=DEM+DFREQ
      SZRQN(J)=1.
      IF(DEM.LE.0.) GO TO 967
      SZRQN(J)=DNUM/DEM
      IF(SZRQN(J).LE.0.) SZRQN(J)=1.
  967 IPOS(J)= IDUES(J)+IONHD(J)
      TMSVD(J)=AMAX1(TMLD(J)+TMNIS-TMISS,0.)
   22 PI(J)=ZWT(1)*PRICE(J)+ZWT(2)*U+ZWT(3)*R/SZRQN(J)+ZWT(4)*TMSVD(J)
      ZSTARM = 0.
       ZS = 0.
      ZR = 0.
      ZU = 0.
      ZT = 0.
      ZMAXS = 0.
      ZSTAR = 0.
      ZMAXT = 0.
      ZMAXU = 0.
      FLAG=0.
      ZMAXR = 0.
  313 DO 100  J=1,N
      QTY1 = DEMFCT(J)
```

```
      IQTY2 = IPOS(J)                                                   GLM00490
      ZMAXS = ZMAXS + PRICE(J)*QTY1                                     GLM00500
      VALDSC=DSCVAL(IQTY2,J)                                            GLM00510
      ZS=VALDSC*PRICE(J)+ZS                                             GLM00520
      ZU=ZU+U*VALDSC                                                    GLM00530
      ZMAXU=ZMAXU+U*QTY1                                                GLM00540
      ZMAXR=ZMAXR + R*QTY1/SZRQN(J)                                     GLM00550
      ZR = ZR + R*VALDSC/SZRQN(J)                                       GLM00560
      ZMAXT=ZMAXT + QTY1*TMSVD(J)                                       GLM00570
      ZT = ZT + VALDSC*TMSVD(J)                                         GLM00580
      ZSTAR = ZSTAR + VALDSC*PI(J)                                      GLM00590
  100 ZSTARM = ZSTARM +(PI(J))*QTY1                                     GLM00600
      WRITE(6,40)     (ZWT(I),I=1,4),ZSTAR,ZSTARM,ZS,ZU,ZR,ZT,ZMAXS,ZMAXGLM00610
     1U,ZMAXR,ZMAXT                                                     GLM00620
   40 FORMAT(7X,'VALUE OF OBJECTIVE FUNCTION EVALUATED AT LEGACY FOR CHOGLM00630
     1ICE OF WEIGHTS',//,14X,F6.4,2X,'FOR SALES',/,14X,F6.4,2X,'FOR UNITGLM00640
     2S ISSUED',/,14X,F6.4,2X,'FOR REQNS ISSUED',/,14X,F6.4,2X,'FOR CUSTGLM00650
     3OMER TIME SAVED',//,6X,'IS',2X,F10.2,'.',//,6X,'IF INFINITE RESOURGLM00660
     4CES WERE AVAILABLE THIS VALUE MIGHT BE INCREASED TO',F11.2,'.',//,GLM00670
     56X,'VALUES OF LEGACY AND GREATEST POSSIBLE VALUE OF THE PURE OBJECGLM00680
     6TIVE FUNCTIONS ARE',//,10X,'SALES',16X,'UNITS',16X,'REQNS',15X,'TIGLM00690
     7ME',/4F20.2,/4F20.2)                                              GLM00700
       ZSTAR = 0.                                                       GLM00710
       ZS = 0.                                                          GLM00720
       ZU = 0.                                                          GLM00730
       ZT = 0.                                                          GLM00740
       ZR = 0.                                                          GLM00750
  312  DO 200 J=1,N                                                     GLM00760
       ITRIAL = IPOS(J) + 1                                             GLM00770
       RUSS= SHADB * PRICE(J)/ PI(J)                                    GLM00780
       D= DEMFCT(J)                                                     GLM00790
       V = DEVFCT(J)                                                    GLM00800
       BILL = ANDTRY(RUSS)                                              GLM00810
       DEB = BILL*DEVFCT(J) + DEMFCT(J)                                 GLM00820
       APOS = IPOS(J)                                                   GLM00830
       IF(DEB.LT.APOS) DEB=APOS                                         GLM00840
       MIKE = DEB + .5                                                  GLM00850
       ITRIAL = MIKE                                                    GLM00860
  201  VALDSC = DSCVAL(ITRIAL,J)                                        GLM00870
       IBUYQ = ITRIAL - IPOS(J)                                         GLM00880
       BUYQ = IBUYQ                                                     GLM00890
       TSTBY = PI(J)* VALDSC - (SHADB * PRICE(J) * BUYQ) - SHADW        GLM00900
       ICE = IPOS(J)                                                    GLM00910
       ALICE = DSCVAL(ICE,J)                                            GLM00920
       TSTNBY = PI(J)* ALICE                                            GLM00930
       IF(TSTBY.GT.TSTNBY) GO TO 204                                    GLM00940
       WINNER = ALICE                                                   GLM00950
       INVNEW(J) = IPOS(J)                                              GLM00960
```

```
 199 ZSTAR= ZSTAR + PI(J)*WINNER                                              GLM00970
     ZS = ZS + WINNER*PRICE(J)                                                GLM00980
     ZU = ZU + WINNER * U                                                     GLM00990
     ZR = ZR + WINNER * R / SZRQN(J)                                          GLM01000
     ZT = ZT + WINNER * TMSVD(J)                                              GLM01010
     GO TO 200                                                                GLM01020
 204 WINNER = VALDSC                                                          GLM01030
     INVNEW(J)=ITRIAL                                                         GLM01040
     NORDER = NORDER + 1                                                      GLM01050
     CSTBUD = CSTBUD +  BUYQ * PRICE(J)                                       GLM01060
     GO TO 199                                                                GLM01070
 200 CONTINUE                                                                 GLM01080
     UBOUND = ZSTAR - (CSTBUD - BUDGET)*SHADB -(NORDER- WRKLD)*SHADW          GLM01090
     WRITE(6,300) BUDGET, WRKLD                                               GLM01100
 300 FORMAT(10X,'BUDGET =',F10.2,10X,'WORKLOAD LIMIT IS',F5.0)                GLM01110
 301 WRITE(6,302) CSTBUD, NORDER, UBOUND                                      GLM01120
 302 FORMAT(5X,'COST IS',F10.2,5X,'NORDER =',I4,5X,'UBOUND =',F10.2)          GLM01130
     WRITE(6,329) ZSTAR, ZS, ZU, ZR, ZT                                       GLM01140
 329 FORMAT(5X,'ZSTAR = ', F8.2,'ZS = ',F8.2,'    ZU =',F8.2,'   ZR           GLM01150
    1=',F8.2,'ZT=',F8.2)                                                      GLM01160
     WRITE(6,303)                                                             GLM01170
 303 FORMAT(3X,'TRY AGAIN?')                                                  GLM01180
     READ(5,304) IANS                                                         GLM01190
 304 FORMAT(A3)                                                               GLM01200
     IF(IANS.EQ.IC2) GO TO 400                                                GLM01210
     ZSTAR = 0.                                                               GLM01220
     ZU = 0.                                                                  GLM01230
     CSTBUD = 0.                                                              GLM01240
     ZS = 0.                                                                  GLM01250
     NORDER = 0.                                                              GLM01260
     ZR = 0.                                                                  GLM01270
     ZT = 0.                                                                  GLM01280
     ZMAXS = 0.                                                               GLM01290
     ZMAXT = 0.                                                               GLM01300
     ZMAXR = 0.                                                               GLM01310
     ZMAXU = 0.                                                               GLM01320
     FLAG = 0.                                                                GLM01330
     ZSTARM = 0.                                                              GLM01340
     WRITE(6,305)                                                             GLM01350
 305 FORMAT(3X,'SAME WEIGHTS?')                                               GLM01360
     READ(5,304) IANS                                                         GLM01370
     IF(IANS.EQ.IC1) GO TO 350                                                GLM01380
     FLAG=1.0                                                                 GLM01390
 310 WRITE(6,307)                                                             GLM01400
 307 FORMAT(3X,'INPUT ZS,ZU,ZR,ZT VIA 4F5.3')                                 GLM01410
     READ(5,306) ZWT(1),ZWT(2),ZWT(3),ZWT(4)                                  GLM01420
 306 FORMAT(4F5.3)                                                            GLM01430
     WRITE(6,308)(ZWT(I),I=1,4)                                               GLM01440
```

```
308 FORMAT(3X,'WEIGHTS ARE',3X,4(F6.3,2X))
    WRITE(6,309)
309 FORMAT(3X,'CORRECT?')
    READ(5,304) IANS
    IF(IANS.EQ.IC2) GO TO 310
    DO  800 J=1,N
800 PI(J)=ZWT(1)*PRICE(J)+ZWT(2)*U+ZWT(3)*R/SZRQN(J)+ZWT(4)*TMSVD(J)
350 WRITE(5,363) SHADB,SHADW
363 FORMAT(3X,'SHADB =',F10.3,5X,'SHADW =',F10.3,/,3X,'SAME MULTIPLIER
   1S?')
    READ(5,304) IANS
    IF(IANS.EQ.IC1) GO TO 312
    WRITE(6,315)
315 FORMAT(3X,'INPUT SHADB AND SHADW VIA 2F10.3')
316 FORMAT(2F10.3)
    READ(5,316) SHADB,SHADW
    GO TO 312
400 WRITE(6,401)
401 FORMAT(3X,'LOOK AT LINE ITEMS?')
    READ(5,304) IANS
    IF(IANS.EQ.IC2) GO TO 500
    WRITE(6,981)
981 FORMAT('    HOW MANY?',/,' XXX')
    READ(5,982)NUMI
982 FORMAT(I3)
    WRITE(6,403)
403 FORMAT('  ITEM  LEGACY  NEW QTY')
    DO 402 J=1,NUMI
    IF(IPOS(J).EQ.INVNEW(J) ) GO TO 402
    WRITE(6,404) J, IPOS(J), INVNEW(J)
404 FORMAT(2X,I4,3X,I5,3X,I5)
402 CONTINUE
    WRITE(8,403)
    DO 405 J=1,N
    WRITE(8,404) J, IPOS(J),INVNEW(J)
405 CONTINUE
500 STOP
    END
    FUNCTION DSCVAL(I,J)
    COMMON DEMFCT(100), DEVFCT(100), IPOS(100)
    K=1
    DSCVAL = 0.
    IF(I.LE.0) GO TO 182
    D = DEMFCT(J)
    V = DEVFCT(J)
180 DSCVAL = DSCVAL + DLTAZJ(K,D,V)
    IF(K.GE.I) GO TO 182
    K=K+1
```

```
      GO TO 180                                                         GLM01930
182   RETURN                                                            GLM01940
      END                                                               GLM01950
      FUNCTION DLTAZJ(I,D,V)                                            GLM01960
      COMMON DEMFCT(100),DEVFCT(100),IPOS(100)                          GLM01970
      SIGMA = 1.25*V                                                    GLM01980
      AMIN = D* .5                                                      GLM01990
      IF(SIGMA.LT.AMIN) SIGMA = AMIN                                    GLM02000
      IF(SIGMA.LT.1.) SIGMA = 1.                                        GLM02010
      AI = I                                                            GLM02020
      X =(AI - D)/SIGMA                                                 GLM02030
      AX = ABS(X)                                                       GLM02040
      T = 1.0/(1.0 + .2316419*AX)                                       GLM02050
      D = 0.3989423 * EXP(-X*X/2.0)                                     GLM02060
      DLTAZJ = 1.0 - D*T*((((1.330274*T - 1.821256)*T + 1.781478)*T     GLM02070
     1 - 0.3565638)*T + 0.3193815)                                      GLM02080
      DLTAZJ = 1.0 - DLTAZJ                                             GLM02090
      IF(X) 1,2,2                                                       GLM02100
1     DLTAZJ= 1.0 - DLTAZJ                                              GLM02110
2     RETURN                                                            GLM02120
      END                                                               GLM02130
      FUNCTION ANDTRY(APROB)                                            GLM02140
      IF(APROB.GT..995)APROB=.995                                       GLM02150
      IF(APROB.LT..005)APROB=.005                                       GLM02160
 7    D = APROB                                                         GLM02170
      IF(D - 0.5) 9,9,8                                                 GLM02180
 8    D = 1.0 - D                                                       GLM02190
 9    T2 = ALOG(1.0 /(D*D))                                             GLM02200
      T = SQRT (T2)                                                     GLM02210
      X = T - (2.515517+0.802853*T+0.010328*T2)/(1.0+1.432788*          GLM02220
     1T+0.189269*T2 + 0.001308*T*T2)                                    GLM02230
      IF(APROB - .5) 10,10,11                                           GLM02240
 10   X = -X                                                            GLM02250
 11   D = 0.3989423*EXP(-X*X/2.0)                                       GLM02260
      X = -X                                                            GLM02270
      ANDTRY = X                                                        GLM02280
 12   RETURN                                                            GLM02290
      END                                                               GLM02300
*/
//SAVAGE JOB (2206,1619,AS42),'SAVAGE'
// EXEC PRTPUNCH
//SYSIN DD *
W4PLOT
OSPLOT
```

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